THE ELECTRIC CIRCUIT

BY

VLADIMIR KARAPETOFF

THE ELECTRIC CIRCUIT

BY

V. KARAPETOFF

ITHACA, NEW YORK

1910

mech eng. dept

PRESS OF
ANDRUS & CHURCH
ITHACA, N. Y.

PREFACE.

This pamphlet, together with the companion pamphlet entitled "The Magnetic Circuit," is intended to give a student in electrical engineering the theoretical elements necessary for calculation of the performance of dynamo-electric machinery and of transmission lines. The advanced student must be taught to treat every electric machine as a particular combination of electric and magnetic circuits, and to base its performance upon the fundamental theoretical relations rather than upon a separate "theory" established for each kind of machinery, as is often done.

The first chapter is devoted to a review of the direct-current circuit, the next four chapters treat of sine-wave alternating-current circuits, and the last two chapters give the fundamental properties of the electrostatic circuit. All the important results and methods are illustrated by numerical problems of which there are over one hundred in the text. The pamphlet is *not* intended for a beginner, but for a student who has had an elementary descriptive course in electrical engineering and some simple laboratory experiments.

The treatment is made as far as possible uniform, so that the student sees analogous relations in the direct-current circuit, in the alternating-current circuit, in the electrostatic circuit, and finally in the magnetic circuit. All matter of purely historical or academic interest, not bearing directly upon the theory of electric machinery, has been left out. An ambitious student will find a more exhaustive treatment in the works mentioned at the end of the pamphlet.

The electrostatic circuit is treated in accordance with the modern conception of elastic displacement of electricity in dielectrics. No use has been made of the action of electric charges at a distance, or of the electrostatic system of units. The volt-ampere-ohm system of units is used for electrostatic calculations, in accordance with Professor Giorgi's ideas (see a paper by Professor Ascoli in Vol. I of the Transactions of the

International Electrical Congress, St. Louis, 1904). Those familiar with Oliver Heaviside's writings will notice his influence upon the author, in particular in Arts. 22 and 23, where an attempt is made at a rational electrostatic nomenclature.

Many thanks are due to the author's friend and colleague, Mr. John F. H. Douglas, instructor in electrical engineering in Sibley College, who read the manuscript and the proofs, checked the answers to the problems, and made many excellent suggestions for the text.

Cornell University, Ithaca, N. Y.
August 1910.

TABLE OF CONTENTS.

CHAPTER I. ELECTRICAL RELATIONS IN DIRECT-CURRENT CIRCUITS.

		PAGE
1.	The Volt, the Ampere, and the Ohm	1
2.	Resistances and Conductances in Series and in Parallel	3
3.	Resistivity and Conductivity	6
4.	Current Density and Electric Intensity	8
5.	Conductors of Variable Cross-section	9

CHAPTER II. REPRESENTATION OF ALTERNATING CURRENTS AND VOLTAGES BY SINE WAVES AND BY VECTORS.

6.	Sinusoidal Voltages and Currents	13
7.	Representation of a Sine-wave by a Vector	17
8.	Addition and Subtraction of Vectors	20

CHAPTER III. POWER IN ALTERNATING-CURRENT CIRCUITS.

9.	Power when Current and Voltage are in Phase.—Effective value of sine wave	25
10.	Power when Current and Voltage are out of Phase.—Wattless and Energy Components.—Form factor	30

CHAPTER IV. REACTANCE AND RESISTANCE IN ALTERNATING-CURRENT CIRCUITS.

11.	Inductance and Reactance	35
12.	Problems on Impedance, and Impedances in Series	37
13.	Impedances in Parallel	38
14.	Equivalent Resistance and Reactance	39
15.	Admittance and its Components—Conductance and Susceptance	41

CHAPTER V. THE USE OF COMPLEX QUANTITIES.

16.	Addition and Subtraction of Vectors in Projections	45
17.	Rotation of Vectors by Ninety Degrees	47
18.	Impedance and Admittance Expressed as Complex Quantities.—Abbreviated notation	50
19.	Power and Phase Displacement Expressed Through Projections of Vectors	55

CHAPTER VI. THE ELECTROSTATIC CIRCUIT.

20.	Physical Concept of the Electrostatic Field	58
21.	Dielectric Flux Density and Electrostatic Stress (Voltage Gradient)	63
22.	Permittivity and Dielectric Strength	67
23.	Permittance, or Electrostatic Capacity	69

CHAPTER VII. THE ELECTROSTATIC CIRCUIT. (Continued.)

24.	Capacity in Alternating-Current Circuits	76
25.	Energy in the Electrostatic Field	78
26.	The Electrostatic Corona	82
27.	Dielectric Hysteresis and Conduction	84
	LITERATURE.	

CHAPTER I.

ELECTRICAL RELATIONS IN DIRECT-CURRENT CIRCUITS.

1. **The Volt, the Ampere, and the Ohm.** The student is supposed to be familiar with Ohm's law, both theoretically and from his laboratory experience. A brief synopsis of the law, given below, is intended merely to refresh the relations in his mind, and to establish a point of view which permits of extending these relations to alternating-current circuits. Moreover, the law is presented in a form applicable to the magnetic and to the dielectric circuits.

When the current in a conductor is steady, its value is proportional to the voltage at the terminals of the conductor. This is an experimental fact, called Ohm's law. Considering the electromotive force as the cause of the current, this law merely states that the effect is proportional to the cause, or

$$e = r \cdot i \qquad (1)$$

where the coefficient of proportionality r is called the resistance of the conductor. When the current is expressed in amperes, and the electro-motive force in volts, the resistance r is measured in units called ohms.

With our present meager knowledge of the true nature of electrical phenomena it is well-nigh impossible to give a clear physical meaning of the quantities under discussion, without resorting to analogies. For instance, the flow of current through a conductor may be compared to the flow of heat through a rod, the voltage being analogous to the difference of temperatures at the ends of the rod, and the electric current to the quantity of heat passing through a cross-section of the rod in unit time (rate of flow of heat). The ratio of the two can be called the resistance of the rod to the passage of heat. Again, the phenomenon of flow of electricity is somewhat analogous to the flow of water through pipes. The hydraulic head may be likened to the voltage, and the rate of discharge of water to the current. With very low velocities, in capillary tubes, the discharge is proportional to the head, so that eq. (1) holds true for the flow of water.

Whatever the reasons which have originally led to the choice of the magnitudes of the ampere, the ohm, and the volt, these units may be considered at present, for all practical and most theoretical purposes, as arbitrary units, same as the foot, the pound, or the meter. Their values have been established by international agreements, whence the name: international electrical units. These units are represented by concrete standards with minutely specified dimensions and properties: The ohm by a column of mercury, the ampere by a silver voltameter, and the volt by standard cells. It is understood, of course, that only two out of the three units need to be standardized, the third being determined either as their product, or their ratio. At this writing, there is some discussion, as to whether the ampere or the volt should be legalized. This discussion is of no immediate importance to the engineer, since both the volt and the ampere are known with an accuracy far exceeding his needs.

The ampere, the volt, and the ohm are connected by simple multipliers (powers of 10) with the absolute electro-magnetic units, or the so-called C.G.S. system of units. It is conceded at present by some prominent physicists that the choice of the units was not quite fortunate, according to our present understanding of the electro-magnetic relations. Since, however, it is too late to change these units, it is better to consider them as arbitrary units, not connected in any way with the magnitudes of the centimeter, the gramme, and the second.

In applying Ohm's law to practical problems it must be remembered that e represents the *net* voltage at the ends of the conductor r. It is necessary to keep this in mind when the circuit contains sources of counter-electromotive forces, such as electric batteries, or motors. Let, for instance, the total resistance of a circuit, connected across the terminals of a generator, be 12 ohms, and let the terminal voltage of the generator be 120 volts. Then the current is equal to 10 amperes, provided that there are no counter-electromotive forces in the circuit. Let, however, the circuit contain a storage battery of say 24 volts, connected so as be charging, that is opposing the applied voltage. The current in the circuit is now only $(120 - 24)/12 = 8$ amps., the value $120 - 24 = 96$ being the *net* voltage in the external circuit. Should the terminals of the battery be reversed, so as

to help the generator voltage, the current would increase to $(120 + 24)/12 = 12$ amps.

2. Resistances and Conductances in Series and in Parallel. When resistances are added in series, the total resistance of the circuit increases. The reason is the same for which an increase in the length of a pipe makes its frictional resistance larger, or increasing the length of a rod makes the passage of heat through it more difficult. The equivalent resistance of two resistances connected in series is equal to their sum. This follows directly from the experimental fact that the current is the same in all parts of the same circuit. Namely, let two conductors r_1 and r_2 be connected in series across a source of voltage e, and let a current i flow through them. Part of the total voltage e is spent in overcoming the resistance of the first conductor, the rest in overcoming that of the second conductor. But, according to Ohm's law, when the conditions are steady, the voltage across the first conductor, $e_1 = i \cdot r_1$; the voltage across the second is $e_2 = i \cdot r_2$. Adding these two equations, gives the total voltage

$$e = e_1 + e_2 = i(r_1 + r_2).$$

An equivalent conductor, r_{eq}, by definition, is a conductor which, with the same total voltage e, allows the same current i to pass through the circuit, as the combination of the two given conductors. Or

$$e = i \cdot r_{eq}.$$

Comparing the two foregoing equations, gives

$$r_{eq} = r_1 + r_2. \qquad (2)$$

The law can be extended to any number of conductors in series, by gradually combining them into groups of two.

When several conductors are connected in parallel, the voltage across them is the quantity common for all the branches, so that we have

$$\left.\begin{array}{l} e = i_1 \cdot r_1 \\ e = i_2 \cdot r_2 \\ \cdots\cdots \\ \cdots\cdots \end{array}\right\} \qquad (3)$$

where i_1, i_2, ... are the currents in the separate branches. The total current flowing through the given system of conductors is equal to the sum of the currents in the separate branches.

This follows from the experimental fact that electricity in its flow behaves like an incompressible fluid, that is to say, the same quantity of it must pass, at each moment, through all the cross-sections of a circuit. This is again analogous to the flow of heat, or of water. Thus, the equivalent conductor, r_{eq}, is determined by the condition

$$e = (i_1 + i_2 + \ldots\ldots) \cdot r_{eq}. \quad \ldots\ldots \quad (4)$$

Substituting the values of i_1, i_2, etc., from (3) into (4) and cancelling e, gives

$$\frac{1}{r_{eq}} = \frac{1}{r_1} + \frac{1}{r_2} + \text{etc.} \quad \ldots\ldots \quad (5)$$

or, in words: when two or more conductors are connected in parallel, the reciprocal of the resistance of the equivalent conductor is equal to the sum of the reciprocals of the resistances of the individual conductors.

When the resistance of one of the conductors in parallel is equal to zero, the corresponding term in eq. (5) becomes infinitely large. The equivalent resistance of the whole combination is in this case equal to zero. This is natural to expect because the conductor of zero resistance short-circuits the other resistances.

Whenever resistances in parallel are to be investigated, their reciprocals enter into formulæ. It is convenient, therefore, to define the reciprocal of a resistance as a separate physical quantity. This is legitimate because, by definition, resistance is merely a coefficient of proportionality between the voltage and the current. Ohm's law can be written in the form

$$i = g \cdot e \quad \ldots\ldots \quad (6)$$

where the new coefficient of proportionality

$$g = 1/r$$

is called the *conductance* of the conductor. The reason for this name is easy to see: the resistance shows how *difficult* it is to force a unit current through a given conductor, while its reciprocal g shows how *easy* it is to produce the same current in the same conductor. Conductances are measured in units called *mhos*, one mho being the reciprocal of one ohm. Hence, a standard resistance, equal to one ohm, represents at the same time a standard conductance of one mho. A resistance of two ohms

has a conductance of one half mho, etc. Increasing the resistance of a winding from 4 to 5 ohms reduces its conductance from .25 to .20 mho.

Equation (5) becomes simply

$$g_{eq} = g_1 + g_2 + \text{etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or the equivalent conductance of several conductors in parallel is equal to the sum of their conductances. As a general rule, in the solution of problems it is convenient to use conductances when conductors are in parallel, and to use resistances, when they are in series. In this way, the necessity for introducing reciprocal expressions into formulae is obviated.

Power converted into heat in a conductor is experimentally found to be proportional to ei. In this respect electricity behaves like a liquid in a pipe. There also the power is proportional to the rate of flow times the difference of pressures at the ends of the pipe. The three units, the volt, the ampere, and the watt have been originally so established, that the coefficient of proportionality is equal to unity, and watts = volts × amperes. If other units are used for power, for instance kilowatts, or horsepower, a coefficient of proportionality is necessary.

If r is known, instead of e, then substituting its value from eq. (1) we find the familiar expression i^2r for the power. If it is desired to eliminate i, the expression for power becomes e^2/r. Using conductance, instead of resistance, in the last two expressions, gives two additional formulae for power:

$$\text{Power} = i^2/g = e^2g. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7a)$$

PROB. 1. Two resistances, $r_1 = 5$ ohm, and $r_2 = 7$ ohm, are connected in series. Resistance r_1 is shunted by a comparatively high resistance, $R_1 = 100$ ohm.; r_2 is shunted by a resistance $R_2 = 50$ ohm. What is the equivalent resistance of the whole combination? Solution:

equivalent conductance of r_1 and R_1 is $.2 + .01 = .21$ mho;
" resistance " " " " " $1/.21 = 4.76$ ohm;
" conductance of r_2 and R_2 is
$.1429 + .0200 = .1629$ mho;
" resistance of r_2 and R_2 is $1/.1629 = 6.14$ ohm.

Ans. $4.76 + 6.14 = 10.90$ ohm.

PROB. 2. If the currents in the shunted resistances R_1 and R_2 in the preceding problem represent pure loss of power, what is the efficiency of the whole arrangement? Solution: Let the voltage across the resistances r_1 and R_1 be e. Then the voltage across r_2 and R_2 is $e_x \times 6.14/4.76 = 1.29\,e$. Hence, the useful power is $e^2/5 + (1.29e)^2/7 = .438\,e^2$ watts. The power lost in the resistances, R_1 and R_2, is $e^2/100 + (1.29e)^2/50 = .0433\,e^2$ watts. Therefore, the efficiency is $.438/(.438 + .0433) = 90$ per cent.

PROB. 3. Four resistances, $r_1 = 1.2$, $r_2 = 1.7$, $r = 25$, and $r_0 = 750$ ohms, are connected as shown in Fig. 1. The generator voltage between the points A and B is 500 v. Determine the current through the resistance r and the voltage across this resistance. NOTE. This combination represents a transmission line of a resistance $r_1 + r_2$, the useful load resistance r, and the leakage

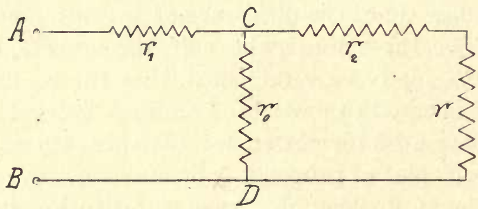

FIG. 1. A series-parallel combination of resistances.

resistance r_0. The problem is of a great importance in alternating-current circuits; see problem 96, Chapter V. Solution: Combine the resistances r_2 and r into one; determine the conductance $1/(r + r_2)$, and combine it with the leakage conductance $1/r_0$. Determine the equivalent resistance between the points C and D, and the total resistance between A and B. Having found the total current, subtract from the generator voltage the voltage drop in the part AC of the line. This will give the voltage across CD, and consequently the value of the leakage current. After this, the drop in r_2 is determined, and thus the voltage across the resistance r is found. Ans. 447.3 volts; 17.88 amps.

3. Resistivity and Conductivity. A cylindrical conductor can be considered as a combination of *unit conductors* in series and in parallel. For instance, a wire 12 m. long and having a cross-

section of 70 sq. mm. can be regarded as composed of $70 \times 12 = 840$ unit conductors, each of one square millimeter cross-section, and one meter long. These unit conductors are first combined into sets of 70 in parallel, and then 12 sets are connected in series. The resistance of such a unit conductor, made of copper, and at a temperature of 0° C., is about .016 ohm. A set of 70 unit conductors in parallel has 70 times less resistance, because the current is offered 70 paths, instead of one; twelve sets connected in series offer twelve times greater resistance than each set. Therefore the resistance of the given conductor is $(.016/70) \times 12 = .002743$ ohm.

Each material is characterized by the resistance of a unit conductor made out of it. The resistance of a unit conductor is called the *resistivity** of the material and is denoted by ρ. Thus, the resistance of a conductor of a length l and cross-section A is

$$r = \rho \cdot l/A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The numerical value of ρ depends upon the units of length and resistance used. A unit conductor may have the cross-section of one square millimeter, and be one meter, or one kilometer long; or it may be a centimeter cube. In the English system it may have the cross-section equal to one circular mil, and be one foot long, or one thousand feet long, etc. The resistance of a unit conductor can be expressed in ohms, megohms, microhms, etc. One megohm = one mililon ohms; one microhm = one millionth part of an ohm. In each case, the unit of resistance, and the units in which the dimensions of l and A are expressed in formula (8), are selected so as to suit the convenience of the user of the formula. For the values of ρ for various materials, and for the effect of temperature, see various handbooks and pocket-books for electrical engineers.

In some cases it is more convenient to use the conductance of the unit conductor, instead of its resistance. The conductance of a unit conductor is called the *electric conductivity* of the material, at a certain temperature; it is equal to the reciprocal of its resistivity. Denoting this conductivity by γ we have: $\gamma = 1/\rho$; the conductance of a conductor having dimensions l and A is

$$g = \gamma \cdot A/l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

* The older name is "specific resistance."

PROB. 4. Resistivity of aluminum is $= 2.66$ microhms per cu. cm.; what is its conductivity per mil-foot? Solution: Resistance of a conductor one foot long and having its cross-section equal to one circular mil is $2.66 \times 10^{-6} \times 197300 \times 30.48 = 16$ ohms; where $197300 = (1000/2.54)^2 \times 4/\pi$ is the conversion factor of square centimeters into circular mils; 30.48 is the ratio of one foot to one centimeter. Ans. Conductivity of aluminum is $= 1/16 = .0625$ mho. per circular mil foot.

4. **Current Density and Electric Intensity.** The use of the coefficients of resistivity and conductivity permits of the following interpretation of Ohm's law, that is convenient in some applications: Substitute the value of g from eq. (9) into (6), and divide the result by A; we get: $i/A = \gamma \cdot e/l$, or

$$U = \gamma F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where

$$U = i/A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and

$$F = e/l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The quantity U is called *current density*, being equal to the current per unit cross-section of the conductor. The quantity F is called *electric intensity*; it represents the voltage drop per unit length of the conductor, or the rate of change of the voltage with the length. Equation (10) states Ohm's law in application to a unit conductor, because U is the current flowing through the unit conductor, F is the voltage across it, and γ is the conductance of the unit conductor. In this form, the relation is independent of the actual dimensions of the conductor, which fact is of advantage in the solution of some problems.*

PROB. 5. What is the voltage drop per kilometer in a copper wire, having a cross-section of 70 sq. mm., and carrying a current

*The treatment in Sections 4 and 5 is intended to prepare the student's mind to a reasoning in terms of the quantities which are of importance in electrostatics and in magnetism. Eq. (10) has its perfect analogs in the electrostatic and in the magnetic circuits: to the current density correspond there the dielectric flux density and the magnetic flux density. The electric intensity is analogous to the electrostatic intensity and to the magnetic intensity; while conductivity finds its analogs in permittivity and in permeability. See Chapter VI of this pamphlet, and the pamphlet on "The Magnetic Circuit."

of 150 amperes? Solution: $U = 150/70 = 2.143$ amps per sq. mm. Conductivity of copper is $\gamma = 57$ mhos. for a unit conductor of one sq. mm. cross-section, and one meter long. Therefore, the electric intensity, or the voltage drop per meter of length, is, according to formula (10), $F = 2.143/57 = .0376$ volt. per meter length. Ans. 37.6 volt./km.

PROB. 6. What is the expression for power converted into heat in a unit conductor? Ans.

$$\text{Power} = F \cdot U = U^2 \rho = U^2/\gamma = \gamma \cdot F^2. \quad . \quad . \quad (13)$$

PROB. 7. What is the amount of power lost in the conductor considered in problem 5? Ans. $(.0376 \times 2.143) \times 70 \times 1000 = 5640$ watts.

PROB. 8. The space available for a rectangular field-coil, on the frame of a generator, is 16×12 cm. for the inside dimensions, and 28×24 cm. for the outside dimensions; the available height is 15 cm. What current density can be allowed in such a coil, if 12 sq. cm. of exposed surface are required per watt loss, in order that the temperature of the coil shall not exceed the safe limit? The space factor of the coil is .55, in other words, 55 per cent of the gross space is occupied by copper, the rest being taken by the air spaces and the insulation. Solution: Exposed surface $= 2(28 + 24) \times 15 = 1560$ sq. cm.; therefore, 130 watts loss can be allowed in the coil. With a space factor of .55 the useful cross-section of copper is $6 \times 15 \times .55 = 49.5$ sq. cm.; the average length of one turn is $= 2(22 + 18) = 80$ cm. Therefore, the coil contains $4950 \times .8 = 3960$ unit conductors, each one meter long and one square millimeter in cross-section. Permissible loss per unit conductor is $130/3960 = .0328$ watts. Equation (13) gives then

$$U = \sqrt{.0328 \times 57} = 1.37 \text{ amps. per sq. mm.}$$

This result is independent of the size of the wire, as long as the space factor remains approximately the same. This example shows the convenience of the concept of unit conductor.

5. Conductors of Variable Cross-section. When the cross-section of a conductor varies along its length, the voltage drop per unit length and the current density are also variable. In places where the cross-section of the conductor is comparatively

large, the resistance per unit length is comparatively small, and *vice versa*. The current density, and the electric intensity, also vary from cross-section to cross-section. The relation between the variable intensity F along the conductor, and the total voltage e at its terminals is no more expressed by the simple relation (12), applicable to the whole conductor. Relation (12) must be now written for an infinitesimal length ds of the conductor, because F can be considered constant only for an infinitesimal length. The definition of F remains the same, namely, F is the rate of variation of voltage per unit length of the conductor. Thus, we have:

$$F = de/ds, \text{ or } de = F \cdot ds \quad . \quad . \quad . \quad . \quad (14)$$

where de is the voltage drop in the element ds of the conductor. Total voltage e at the terminals of the conductor is equal to the sum of these infinitesimal drops, or

$$e = \int_0^l F \cdot ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Eq. (15) is usually expressed in words by saying that *voltage is the line integral of electric intensity*.

A clear understanding of the relations (14) and (15) is of paramount importance in the study of electro-static and magnetic phenomena; therefore eqs. (14) and (15) are further expounded below, by resorting to the analogies used before. In the case of flow of heat, F corresponds to the rate of change in temperature per unit length of the rod, while e represents the total difference of temperatures at the ends of the rod. Eq. (15) merely expresses, that by taking the rate at a certain point and multiplying it by a very short element of the length of the rod, the actual difference of temperatures at the ends of this element is obtained. Thus, let, for instance, the drop in temperature at some point of the rod vary at a rate of 2.5 degrees Centigrade per meter length. Then the actual drop upon a very short element, say 0.1 mm., is $2.5 \times .0001 = .00025°$ C. The element of length must be small, because, by supposition, the cross-section of the rod is variable, and the rate of drop is consequently variable. For a short length the variable quantities can be assumed constant, or, more correctly, average values can be used.

Similarly, in a pipe of variable cross-section the rate of loss of head per unit length is variable, so that it is only possible to speak of this rate F at a point. Total loss of pressure, or the head e, is obtained by summing up the small losses of head on infinitesimal elements of the pipe. The loss of pressure for a length ds is $F\,ds$; total head e is the integral of this expression over the total length of the pipe. This is expressed mathematically by eq. (15).

The following examples illustrate some methods of dealing with conductors of non-uniform cross-section.

PROB. 9. Calculate the resistance of a cylindrical layer of mercury MM (Fig. 2) of height $h = 5$ cm., between two concentric cylindrical electrodes A_1 and A_2, the radii of the surfaces of contact being $a_1 = 10$ cm. and $a_2 = 18$ cm. Resistivity of mercury is $\rho = 95$ microhms per cu. cm. Solution: Take an infinitesimal layer of the mercury, between the radii x and $x + dx$; the resistance of this layer, according to eq. (8), is

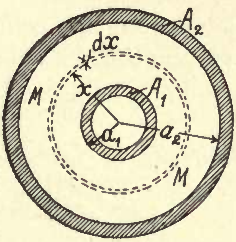

FIG. 2. Flow of current between two concentric electrodes.

$$dr = \rho \cdot dx/2\pi xh.$$

The resistances of all the infinitesimal concentric layers are in series, so that integrating this expression between the limits a_1 and a_2 gives

$$r = \rho/2\pi h. \text{ Ln } a_2/a_1. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Ans. $r = 1.775$ microhms.*

PROB. 10. When a current $i = 10.000$ amps. flows through the mercury in the preceding problem, what is the amount of heat generated per second per cu. cm. of mercury, at both electrodes? Solution: Current density at the inner electrode is $10.000/(2\pi \times 10 \times 5) = 31.8$ amps. per sq. cm. From eq. (13) we find therefore: loss of power $= (31.8)^2 \times 95 \times 10^{-6} = .0958$ watts per cu. cm. The heat loss at the outer electrode is $.096 \times (10/18)^2 = .0295$ watts per cu. cm.

PROB. 11. What is the curve of electric intensity in the arrangement given in problems 9 and 10, and what are the limit-

* Electrostatic capacity between two concentric cylinders is obtained by means of an identical reasoning; see problem 121, in Chapter VI.

ing values of F? Solution: From eq. (10), substituting $1/\rho$ for γ, we have: $F = \rho U$. But in our case U is $= i/2\pi xh$; hence, $F = \rho i/2\pi h \cdot 1/x$. This is the equation of an equilateral hyperbola, if F be plotted to x as abscissae. The result could have been forseen, because the cross-section of the path of the current varies inversely as the distance from the center. Therefore, the current density and the electric intensity must vary inversely as the distance from the center. The value of F at the inner electrode is 3.02 milli-volts per cm. length; at the outer electrode $F = 1.677$ milli-volts per cm.

PROB. 12. A lead-covered cable, consisting of a solid circular conductor of A sq. mm. in cross-section, is insulated with a layer of rubber a mm. thick, between the conductor and the sheathing. What is the insulation resistance of l kilometers of such a cable, if the resistivity of rubber is γ megohms per centimeter cube? Ans. $\rho \times 10^{-5}/2\pi l \cdot \mathrm{Ln}\,(1 + 1.772a/\sqrt{A})$ megohms, according to eq. (16).

PROB. 13. Show that by increasing the thickness of insulation in the preceding problem twice, the insulation resistance is increased less than twice.

PROB. 14. Current is flowing across a hemispherical shell of metal along its radial lines. Express the resistance of the shell as a function of its radii b_1 and b_2, and the conductivity c of the material. Ans. $(b_2 - b_1)/(2\pi c\, b_1\, b_2)$.

PROB. 15. A non-linear irregular conductor, made of homogeneous material, has the current density U and the electric intensity F varying from point to point in magnitude and direction. What is the general expression for the power lost by Joulean heat? Ans. According to eq. (13),

$$\text{Power} = \int F \cdot U dv = \frac{1}{\gamma}\int U^2 dv = \gamma \int F^2 dv \quad . \quad . \quad (17)$$

where dv is the element of volume to which F and U refer, and the integration is extended over the whole volume of the conductor. The volume dv must be taken as a cylinder or parallelopiped, the length of which is in the direction of flow of current, and the cross-section of which is perpenducular to this flow.

CHAPTER II.

REPRESENTATION OF ALTERNATING CURRENTS AND VOLTAGES BY SINE WAVES AND BY VECTORS.

6. Sinusoidal Voltages and Currents. A large proportion of electric power used for lighting, industrial purposes, and traction, is generated in the form of alternating currents. Some of the advantages of the alternating current over the direct current are: (1) Alternating power can be easily converted into power at a higher or at a lower voltage, thus making possible the transmission of power over long distances. (2) The generation of alternating currents is simpler than that of direct currents. The latter require the use of a commutator, which is liable to give troubles in operation. (3) By combining two or three alternating-current circuits into the so-called polyphase systems it is possible to convert electric power into mechanical by motors of simple and rugged construction (induction motors and synchronous motors).

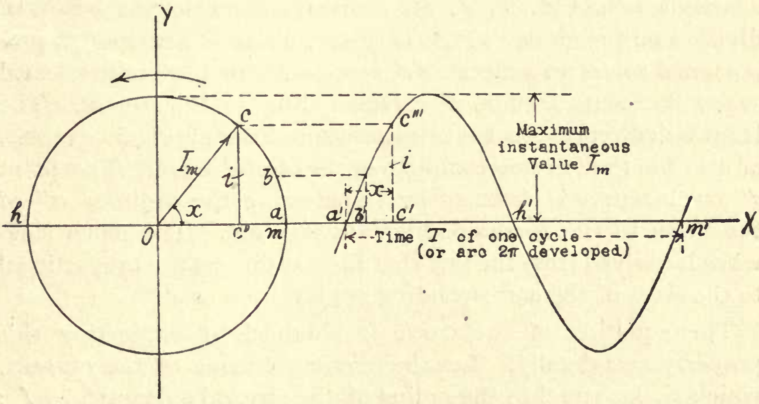

FIG. 3. An alternating current represented by a sine wave.

Alternating voltages, generated by commercial alternators, are more or less irregular in shape, but for most engineering calculations it is accurate enough to assume them to vary with the time according to the sine law (Fig. 3). This assumption

simplifies the theory and calculations greatly, and at the same time gives comparable results, the same referring to a standard shape of the voltage and current curves, instead of particular forms in a specific problem. Moreover, if the curve of a voltage or current differs greatly from the sine-wave, it can be resolved into a series of sine-waves of different frequencies, so that even then the sine-wave remains the fundamental form (see Karapetoff's Experimental Electrical Engineering, under "wave analysis"). Fig. 3 shows the well-known construction of a sine-wave, instantaneous values of the voltage or the current being represented as ordinates, against time as abscissae. Instead of actual time in seconds, the curve can be plotted against some other quantity proportional to time, for instance, fractions of a complete period, or angular positions of the field pole of the alternator with respect to a conductor in which the electro-motive force under consideration is induced.

Draw a circle with the radius equal to the maximum value of alternating current (or voltage) the wave of which is desired to construct. Divide the circle into a certain number of equal or unequal parts, such as ab, bc, etc., and mark on the axis of abscissae points a', b', c', etc. corresponding to the points of division on the circle. That is to say, $a'c'$ is either equal or proportional to ac; in general, $a'c'$ represents, to a certain selected scale, the central angle x corresponding to the arc ac. The length $a'm'$ represents to the same scale the angle of 360 degrees, or also the time of one complete cycle of the wave. The point c''' on the curve is obtained by transferring the ordinate cc'' of the circle to the corresponding abscissa $a'c'$. The name sine-wave is derived from the fact that these ordinates are proportional to the sines of the corresponding angles, such as x.

The equation of the curve is obtained by expressing this property analytically. Let the maximum value of the current, which is also equal to the radius of the circle, be denoted by I_m; we have then from the triangle Occ''

$$i = I_m \sin x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where the ordinate $i = cc'' = c'c'''$ represents the instantaneous value of the alternating current, at the moment of time corre-

sponding to the angle x. The variable angle x is proportional to time, because the radius Oc which generates the sine wave revolves at a uniform speed. Let time t be counted from the position Oa of this radius, and let $T = a'm'$ be the interval of time necessary to complete one revolution of the radius, or the time of one complete *cycle* of the alternating wave. When time $t = 0$, $x = 0$; when $t = T$, $x = 2\pi$. Therefore, in general, $x = 2\pi t/T$; because this expression satisfies the foregoing conditions.

We have then:

$$i = I_m \sin 2\pi t/T. \qquad (19)$$

For the values of $t = 0$, $T/2$, T, $3T/2$, etc., $i = 0$, as it ought to be, because at these moments the current changes from its positive to negative values or *vice versa*. At $t = T/4$, $3T/4$, $5T/4$, etc., we have $i = \pm I_m$; at these moments the current reaches its positive and negative maxima. Eq. (18) is used when the sine-wave is plotted against values of angle x as abscissae. Eq. (19) gives the same curve referred to time t as abscissae.

In practice, the rapidity with which currents and voltages alternate is not denoted by the fraction of a second T during which a cycle is completed, but, in a more convenient manner, by the number of complete cycles in a second. Thus, instead of saying that an alternator generates current which completes a cycle within 1/60th of a second, it is customary to say that the frequency of the current is 60 cycles per second. Denoting the frequency in cycles per second by f we have

$$f = 1/T \qquad (20)$$

and consequently

$$i = I_m \sin 2\pi f t. \qquad (21)$$

This is the usual expression for an alternating current of frequency f periods per second. Analogously, for an alternating voltage we have:

$$e = E_m \sin 2\pi f t \qquad (22)$$

where E_m is the maximum instantaneous value, also called the *amplitude*, and e is the instantaneous value at the moment t.

In numerical calculations, and in drawing sine waves, the values of ordinates for various values of x or t are obtained either graphically, as shown in Fig. 3, or from tables of sines. For approximate calculations, values of sines can be taken from a slide-rule. In the problems 16 to 20, that follow, the student is advised to become familiar with all the three methods of getting values of sines.

PROB. 16. An alternating current fluctuates according to the sine law between the values $\pm I_m = 75$ amps., making $n = 6000$ alternations per minute (3000 positive and as many negative ones). Draw a curve of instantaneous values of this current; mark on the axis of abscissae: time t in thousandths of a second, angles x in degrees, and the same angles in radians.

PROB. 17. What is the frequency of the current in the preceding problem, in cycles per second? Ans. $f = 50$.

PROB. 18. Plot on the same curve sheet with the curve obtained in problem 16 the sine-wave of a current the frequency of which is three times higher, the amplitude $I_m = 52$ amps. The curve is to be at its maximum when the first curve is at a maximum.

PROB. 19. Supplement the preceding curves by one the frequency of which $f = 50$ cycles per second, the amplitude $I_m' = 63$ amps., and which reaches its positive maxima at moments in which the first curve passes through zero. Show that with these specifications two distinct curves can be drawn.

PROB. 20. Draw on the same curve sheet with the preceding curves a sine-wave representing an alternating current of the same frequency, $f = 50$ periods per second, and which is determined by the following conditions: Its amplitude $I_m'' = 120$ amps.; it passes through zero and begins to increase in the positive direction 1/600 of a second *later* than the first curve. Thus, it passes through zero at an angular distance, $\phi = 30$ degrees, to the right of the first curve. In the usual engineering language, the first current is said to *lead* the second by 30 degrees, or the second to *lag* behind the first by the same amount. The angle ϕ is called the *angle of phase difference* between the two currents.

PROB. 21. The current mentioned in problem 16 is generated by a twelve-pole alternator, that in problem 18 by a fourteen-pole machine. At what speeds must these machines be driven in order to give the required frequencies? Ans. 500 and 1285 r.p.m.

PROB. 22. Express the currents given in problems 16 to 20 by equations of the form of equation (18). Ans. $i = 75 \sin x$; $i = -52 \sin 3x$; $i = \pm 63 \cos x$; $i = 120 \sin (x - 30°)$.

PROB. 23. The angle x in the answers to the preceding problem is expressed in degrees; rewrite the equations so as to have x expressed in radians, and in fractions of a cycle. Also represent the currents as functions of time t.

PROB. 24. Express by equations similar to equation (22) the following sinusoidal voltages of frequency f: (a) Amplitude E_m volts. (b) Amplitude E_m' volts, lagging α degrees with respect to the first curve. (c) Amplitude E_m'' volts, leading the second curve by ϕ radians. (d) Amplitude E_m''' volts, lagging $1/n$th of a cycle with respect to curve (a).

PROB. 25. The voltages required in the preceding problem are induced by four identical alternators, having p poles each, and coupled together. By what geometrical angles must the revolving or the stationary parts be displaced in order to give the required differences in phase?

7. Representation of a Sine-wave by a Vector. The foregoing problems make clear that all sine-waves are different from one another in three respects only, namely: (1) In amplitude; (2) in frequency; (3) in relative phase position. In most practical cases, all the currents and voltages entering into a problem are of the same frequency, so that they differ from each other solely in their amplitudes and phase positions. It is not necessary in such cases actually to draw sine-waves, or even to write their equations: It is sufficient to indicate the radii generating these curves (Fig. 4), in their true magnitudes and *relative* positions. The rotating radius, at different instants, gives by its vertical projection to scale the magnitude of the alternating current or voltage at that instant. The initial position of the radius, at the instant we choose to begin counting time, will therefore represent graphically the alternating quantity. These radii can be either actually drawn and problems solved graphically, or the

radii can be expressed by their projections on any pair of co-ordinate axes, and treated analytically. This chapter is devoted to the graphical solution of alternating-current problems; the analytical treatment by projections is taken up in the following chapters.

Fig. 4 shows the two sine-waves given in problems 16 and 20, and the two radii generating these two waves. It will be seen

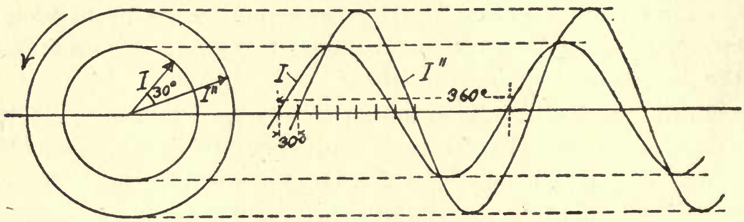

FIG. 4. Two alternating currents displaced in phase by 30 degrees.

that the angle between the radii is 30 degrees, corresponding to the phase displacement between the two waves. In Fig. 3, the curve is generated by the radius revolving counter-clock-wise; the radius representing the second curve in Fig. 4 is shown lagging behind the first radius in accordance with this assumption. The *absolute* position of the radii is irrelevant, because they are revolving all the time. It is their relative position which is permanent and which determines the relative position of the sine-waves. The moment, from which time is counted, is arbitrary in most problems; hence, one of the radii can be drawn in any desired position. Then, all other radii in the same problem become definite by their phase displacements with respect to this "reference" radius.

It must be clearly understood that the foregoing representation by vectors is true only when all the vectors are revolving at the same speed, that is to say, when we are dealing with alternating quantities of the same frequency. When, however, currents and voltages of different frequencies enter into a problem, the angle between the vectors varies all the time, and it is necessary to introduce an arbitrary zero of time for reference. In general, graphical solution is unsuitable for such problems.

In mathematics and physics, a quantity which has not only a magnitude, but also a definite direction in space or in a plane, is called a *vector*. Thus, for instance, a force in mechanics is a vector quantity, while a volume is not a vector quantity. The radii which represent sine-waves have both magnitude and direction in a plane. It is proper, therefore, to call them vectors. While the direction of the first vector is usually arbitrary, once it is selected, the directions of all other radii become altogether definite, so that with this limitation radii in alternating-current problems have definite directions and can be called vectors. While they must be imagined as revolving when generating the corresponding sine-waves, yet they revolve as a system, all together. The required relations always depend upon the relative positions of the radii, so that the fact that they are revolving can be altogether disregarded, and the radii considered as ordinary stationary vectors.

PROB. 26. Draw in Fig. 4 the vectors of the sine-waves of problem 19.

PROB. 27. A single-phase alternator has a terminal voltage such that its maximum instantaneous value E_m is equal to 16 kilovolts. The maximum value of the current supplied by the machine is $I_m = 325$ amps. The character of the load is such that the current wave lags behind the voltage wave by an angle $\phi = 37$ degrees. Assuming both the voltage and the current to vary according to the sine law, represent the foregoing conditions by two vectors.

PROB. 28. Draw a vector diagram showing the phase voltages (star voltages) of a 25 cycle three-phase system,* the amplitude of each voltage being equal to 7235 volts, and displaced in phase by 120 degrees with respect to the other two voltages. The current in the first phase is equal to 30 amps. and lags behind the corresponding phase voltage by 1/6th of a cycle. The current in the second phase is 47 amps. and leads its voltage by 18 degrees. The current in the third phase is equal to 72 amps., and lags behind the corresponding phase voltage by .004 of a second.

*See the chapter on the "Electrical Relations in Polyphase Systems" in the author's "Experimental Electrical Engineering."

NOTE. The student is supposed to draw the vectors in problems 26 to 28 to represent the amplitudes of the corresponding alternating waves. It is customary in practice to draw vectors of the *effective* values of voltages and currents, and not of the amplitudes; for sine-waves the effective value is equal to the amplitude divided by $\sqrt{2}$ (see Chapter III). The advantage of using effective values, instead of amplitudes, is explained at the end of Sec. 9 below. The difference is not essential for our present purposes. The use of effective values would merely change the scale to which the vectors are drawn, which scale is arbitrary anyway.

8. Addition and Subtraction of Vectors. There are many occasions when alternating currents and voltages have to be added together, and subtracted from one another. For instance, when two or more alternators are working in parallel the total current delivered to the bus-bars of the station is equal to the sum of the currents supplied by each machine. Or, to find the voltage at the receiving end of a transmission line the voltage drop in the line is subtracted from the generator voltage. When the component quantities vary according to the sine law the resultant quantity is also a sine curve. This curve can be found graphically, by adding the given curves point by point, or by adding their equations; also by adding the vectors of these curves. All these metods are explained and illustrated below.

PROB. 29. The currents given in problems 16 and 20 are generated by two alternators working in parallel. Find the curve of the total current by adding the ordinates of the curves point by point. Measure the amplitude and the phase position of the resultant current, and write down its equation. Ans. Approximately $189 \sin (x - 18° 30')$, in amperes.

It can be proved analytically that the sum of two sine-waves of the same frequency is also a sine wave, of the same frequency. Therefore, it is not necessary to add sine waves point by point; the equation of the resultant wave can be derived directly from the equations of the component curves. Let one wave be represented by eq. (18); the other wave, which we shall assume to lead the first by an angle ϕ, can be represented by the equation $i' = I_m' \sin (x + \phi)$. The sum of their ordinates is

$$i + i' = (I_m + I_m' \cos \phi) \sin x + I_m' \sin \phi \cdot \cos x,$$

where the constant terms multiplied by sin x and by cos x are collected together. The form of the equation suggests the sine of the sum of two angles. We denote therefore

$$\left.\begin{aligned} I_m + I_m' \cos \phi &= I_{eq} \cos \phi_{eq} \\ I_m' \sin \phi &= I_{eq} \sin \phi_{eq} \end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (23)$$

and obtain

$$i_{eq} = i + i' = I_{eq} \sin (x + \phi_{eq}),$$

where the subscript "equivalent" refers to the amplitude and the phase of the resultant wave. The last equation shows that the resultant current i_{eq} varies according to the sine law, and has the same frequency as the component curves, because the variable angle $x = 2\pi ft$ is the same as in the component waves.

PROB. 30. Find analytically the equation of the resultant wave, required in problem 29. Solution: The equations of the given currents are: 75 sin x and 120 sin $(x - 30°)$, so that eqs. (23) become:

$$\left.\begin{aligned} I_{eq} \cos \phi_{eq} &= 75 + 120 \times \tfrac{1}{2}\sqrt{3} = 178.92 \text{ amps.}; \\ I_{eq} \sin \phi_{eq} &= -\ 120 \times 1/2 = -\ 60 \text{ amps.} \end{aligned}\right\} \quad . \quad . \qquad (23a)$$

Dividing the second equation by the first, gives: $\tan \phi_{eq} = -.3353$, or $\phi_{eq} = -18° 32'$. Consequently, $I_{eq} = -60/\sin \phi_{eq} = 188.8$ amps. Ans. $188.8 \sin (x - 18° 32')$, in amperes.

PROB. 31. Solve the preceding problem without the use of eqs. (23), simply on the basis of the theorem proven above that the sum or the difference of two sine waves is also a sine wave. Solution:

$$I_{eq} \sin (x + \phi_{eq}) = 75 \sin x + 120 \sin (x - 30°).$$

This equation is true for any instant, or for any value of x. It contains two unknown quantities, the amplitude and the phase position of the resultant curve. It is necessary, therefore, to apply this equation to two particular moments of time, in order to obtain two equations with two unknown quantities. It is most convenient in this particular case to choose $x = \pi/2$ and $x = 0$. Substituting these values, eqs. (23a) are obtained. This method is preferable in the solution of practical problems, because it is not necessary to remember eqs. (23).

The foregoing examples show that adding alternating currents either point by point, or analytically, is a somewhat tedious operation. It can be dispensed with because of the fact that the resultant current is always sinusoidal, and therefore can also be represented by a vector. The problem is reduced simply to finding the vector of the resultant wave, knowing the vectors of the component waves in their magnitude and position. Any ordinate of the resultant wave must be equal to the sum of the correspond-

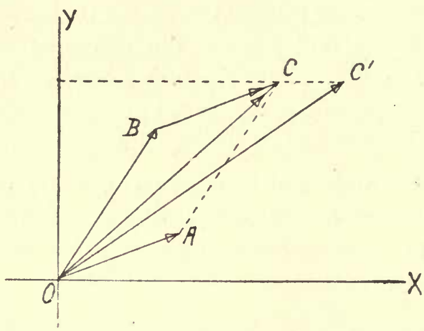

FIG. 5. Addition of vectors.

ing ordinates of the component waves. Hence, the vector of the resultant wave must satisfy the condition that its projection on the *Y*-axis (Fig. 5) be equal to the sum of the projections of the component vectors on the same axis. This condition must be fulfilled at all instants of time, that is to say, when the three vectors are rotated. To satisfy this requirement the resultant vector must be the closing side of the triangle constructed on the other two vectors.

Let *OA* and *OB* be the given vectors to be added together. From the end *B* of the vector *OB* draw a line *BC* equal and parallel to *OA*. Connecting *O* to *C* gives the resultant vector *OC*, in magnitude and position. It is easy to see from the figure that the projection of *OC* on the *Y*-axis is equal to the sum of the projections of *OB* and *BC* upon the same axis. But *BC* is equal and parallel to *OA*, so that the projection of *OC* on the vertical axis is equal to the sum of the projections of the given vectors on the same axis. This construction holds true for any instant. By drawing *AC*, the parallelogram *OBCA* is completed, and it

will be seen that the construction is identical with that of finding the resultant of two forces in mechanics. However, in practical applications it is not necessary to complete the parallelogram, because the resultant is perfectly determined from the triangle OBC. The resultant of two vectors obtained in this way is called their *geometric sum.*

If the triangle of the vectors were not closed, the condition of the sum of projections might be satisfied for one particular instant of the cycle, but not be satisfied for any other instant. Thus, for instance, assuming line OC' to be the resultant vector, we see that its projection on axis OY is equal to the sum of the projections of OB and BC on the same axis for the given moment of time; but the condition is not fulfilled when the figure rotates.

The rule for subtraction of vectors follows immediately from the preceding rule, because to subtract a vector means to add a vector with the opposite sign. Thus, let it be required to subtract vector OA from OB (Fig. 6); this means to subtract the voltage wave represented by OA from that represented by OB. From the end B of OB draw vector BC equal and *opposite* to OA. The resultant, OC, represents the difference of the two given vectors, in direction and magnitude, and thus determines the sine wave of the resultant voltage. If it were required to subtract OB from OA, it would be necessary to draw AC' equal and opposite to OB. The resultant is OC'; it is equal and opposite to the former resultant OC. This is in accord with the general algebraic rule that $A - B = -(B - A)$. By reference to Fig. 5 it will be seen that in the case of addition the order of the vectors does not affect the value or the direction of the resultant. This is also according to the algebraic rule that $A + B = B + A$.

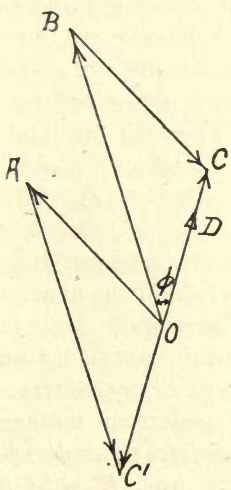

FIG. 6. Subtraction of vectors.

The preceding results with regard to the addition and subtraction of vectors can be summed up in the following rule: *Relations which are true* algebraically *for instantaneous values of sinusoidal currents and voltages hold true* geometrically *for the vectors of these quantities.* It is customary to provide vectors of currents with triangular arrows, as in Fig. 5; vectors of voltages are usually distinguished by pointed arrows, as in Fig. 6. This distinction enables one to see directly from the diagram whether a vector represents a current or a voltage, without reference to the text.

PROB. 32. Construct the vector of the resultant current required in problems 29 and 30, and check its magnitude and phase position with those obtained analytically.

PROB. 33. Two alternators, with the same number of poles, are coupled together so as to give voltages differing in phase by $\alpha = 27$ degrees, the voltage of the second machine leading that of the first by this angle. The first alternator generates voltage $E_m = 2300$ volt., the second $E_m' = 1800$ volts. The two machines are connected electrically in series. Find graphically the vector of the resultant voltage in its magnitude and phase position. Find also the vector of the resultant voltage when the terminals of one of the machines are reversed. Ans. (1) 3988 v., leading E_m by $11° 49'$; (2) 1074 v., lagging behind E_m by $49° 32'$.

PROB. 34. An alternator, the terminal voltage of which is $E = 6600$ volt., supplies a load through a transmission line. The conditions are such that the current lags behind the generator voltage by an angle $\phi = 35$ degrees. The voltage drop in the line is $\epsilon = 540$ volt., leading the current in phase by an angle $\alpha = 67$ degrees. Find the receiver voltage E', by subtracting the voltage drop in the line from the generator voltage (geometrically); also determine the phase displacement ϕ' between the receiver voltage and the current. Ans. $E' = 6149$ volt. $\phi' = 32° 20'$.

CHAPTER III.

POWER IN ALTERNATING-CURRENT CIRCUITS.

9. **Power when Current and Voltage are in Phase.** Let a resistance r be connected across the terminals of an alternator, the voltage at the terminals varying according to the sine law. The current through the resistance also varies according to the sine law, because Ohm's law holds true for any moment of time, so that the curve of current is in phase with that of the voltage. If the equation of the voltage wave is $e = E_m \sin x$, the equation of the current is $i = E_m/r \cdot \sin x$. Graphically, the current and the voltage are represented by two vectors of different length, but in the same direction; for instance, like OC and OD in Fig. 6.

Divide the time T of one cycle into a large number of small intervals Δt. Then the amount of energy delivered to the resistance r and converted into Joulean heat during one of such intervals varies with the time-position of the interval in the cycle; in other words, with the instantaneous values of the voltage and the current. This energy is practically equal to zero, when the current and the voltage have values near zero, and it reaches a maximum with them. However, the energy never becomes negative, because whether the current flows in one direction, or in the opposite, the heat liberated, $i^2 r \cdot \Delta t$, is always positive.

It is customary to speak of the *rate of liberation of energy per unit time*, instead of the actual amounts of energy during very short or infinitesimal intervals of time. It must be understood, of course, that with alternating currents the rate is variable, and, strictly speaking, applies only to one moment of time. Let the instantaneous value of the current at a certain instant t be i. If this current remained constant for one second, the energy liberated would be equal to $i^2 r$. As a matter of fact, this current can be considered constant only during the infinitesimal element of time dt, so that the energy liberated during this element is $i^2 r \cdot dt$. Nevertheless, it is proper to say that at the instant under consideration the energy is liberated *at the rate* equal to $i^2 r$ per unit time, because $i^2 r \cdot dt/dt = i^2 r$. This is ana-

logous to speaking of the instantaneous speed of a body during a period of acceleration or retardation. The speed varies from instant to instant, so that to say that the speed is v at a certain instant merely means that, *if* the body continued to move at this velocity for one second, it *would* cover a space equal to v.

The rate at which energy is liberated, or the energy per unit time, is called *power*. Thus, it is permissible to speak of instantaneous power in the sense in which we speak of instantaneous values of speed in a non-uniform motion. The instantaneous power indicates the amount of energy which *would* be developed per second, *if* the current suddenly became constant.

The total energy liberated in the form of heat during one complete cycle is

$$W = \int_0^T i^2 r \cdot dt = r \int_0^T i^2 \cdot dt. \quad . \quad . \quad . \quad (24)$$

Another expression for power in a direct-current circuit is ei. In an alternating-current circuit, the voltage and the current can be assumed constant during an infinitesimal interval of time dt. Therefore, the instantaneous energy is $ei \cdot dt$, and the energy developed during one cycle is

$$W = \int_0^T ei \cdot dt. \quad . \quad . \quad . \quad . \quad . \quad (24a)$$

PROB. 35. The current given in problem 16 flows through a resistance of 10 ohms. Plot curves of instantaneous values of voltage and power. Ans. $E_m = 750$ volt.; max. power $= 56.25$ kw.

PROB. 36. Determine total energy liberated per cycle in the preceding problem by integrating graphically the curve of power. Ans. 562.5 joules (watt-seconds).

PROB. 37. Prove analytically that the curve of power, obtained in problem 35, is a sine wave of double frequency, tangent to the axis of time. Proof: The equation of the curve is $P = I_m^2 r \cdot \sin^2 x$. But from trigonometry

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x.$$

Substituting this value of $\sin^2 x$ into the expression for P we get:

$$P = \tfrac{1}{2} I_m^2 r - \tfrac{1}{2} I_m^2 r \cdot \cos 2x.$$

The first term is constant, the second represents a sine wave of double frequency, because $2x = 2\pi (2f) t$. The first term is always larger than the second, so that P is always positive, and the whole curve lies above the axis of abscissae. Only when $2x$ is a multiple of 2π the second term becomes equal to the first and $P = 0$. At these points the curve is tangent to the axis of abscissae.

PROB. 38. Integrate expression (24) for the energy per cycle when the current is expressed by eq. (18). Solution:

$$\int_0^T i^2 r dt = I_m{}^2 r \int_0^T \sin^2 x \cdot dt.$$

But $x = 2\pi t/T$; consequently

$$dt = T/2\pi \cdot dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

When $t = 0$, $x = 0$; when $t = T$, $x = 2\pi$. We have therefore, changing the limits of integration:

$$W = I^2_m \cdot T/2\pi \cdot \int_0^{2\pi} \sin^2 x \cdot dx.$$

This integral is best calculated by observing that its value remains the same if cosine is substituted for the sine. This is because integration is a summation, and in summing up sines or cosines over 2π one goes through the same values, only in a different order. Thus, we have:

$$\int_0^{2\pi} \sin^2 x \cdot dx = Q; \int_0^{2\pi} \cos^2 x \cdot dx = Q,$$

where Q is the unknown value of both integrals. Adding these integrals together, and remembering that $\sin^2 x + \cos^2 x = 1$, we obtain:

$$\int_0^{2\pi} dx = 2\pi = 2Q,$$

or, $Q = \pi$. Hence,

$$\int_0^{2\pi} \sin^2 x \cdot dx = \pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Consequently,

$$W = \tfrac{1}{2} I^2_m r \cdot T. \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

PROB. 39. Perform the integration required in the preceding problem by replacing $\sin^2 x$ by the cosine of the double angle; see problem 37.

In practice, *the average rate* at which energy is delivered is of interest, in other words, the average value of the variable power. This is analogous to using in calculations the average speed of a machine, when the actual speed varies within certain limits. This average power is found by dividing the total energy developed during one cycle by the time T of one cycle. When the current varies according to the sine law the total energy per cycle is expressed by eq. (27). Dividing both sides by T we find that the average power

$$P = \tfrac{1}{2} I^2_{\mathrm{m}} r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Had we started with eq. (24a) instead of eq. (24) we would have arrived at the result

$$P = \tfrac{1}{2} E_{\mathrm{m}} I_{\mathrm{m}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28a)$$

Expressions (28) and (28a) are identical, because $E_{\mathrm{m}} = rI_{\mathrm{m}}$, according to Ohm's law. It is convenient to use in eq. (28) a new value of the current, $I = I_{\mathrm{m}}/\sqrt{2}$, instead of I_{m}, because then the expression for power becomes identical with that used with direct currents, namely

$$P = I^2 r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

A constant current I so chosen that its square is equal to the mean value of the squares of instantaneous values of a given alternating current, is called *the effective value of the alternating current.* It is seen from formulae (28) and (29) that for sinusoidal currents the effective value is equal to the amplitude divided by the square root of two, or

$$I = I_{\mathrm{m}}/\sqrt{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

The effective value, as defined above, gives the same average power and the same total energy during one cycle as the actual alternating current which it represents. We have

$$I^2 r \cdot T = r\int_0^T i^2 \cdot dt$$

or

$$I^2 = \int_0^T i^2 dt / T. \quad . \quad . \quad . \quad . \quad . \quad . \quad (30a)$$

The last equation expresses in mathematical language that I^2 is the average value of i^2 over a cycle. Taking the square root of both sides of this equation, we can also define the effective value I as *the square root of the mean square of instantaneous values*. This definition is true for any form of alternating waves.*

PROB. 40. Figure out the effective value of the current in problem 35, and show that the expression $I^2 r$ is equal to the average ordinate of the power curve, plotted before.

Power is also expressed in direct-current circuits by e^2/r. Analogously, total energy per cycle of an alternating-current circuit can be expressed by

$$W = \int_0^T e^2/r \cdot dt,$$

where e is the instantaneous value of the voltage. It will be remembered, however, that this discussion is limited to the case where the circuit consists only of resistance r, and contains no counter-electromotive forces, caused by inductance, capacity, or motors. Only under this assumption the relation $i = e/r$ holds true for each instant, and consequently e^2/r represents the rate of dissipation of energy, (see the end of Sec. 1).

Here again, it is convenient to introduce the effective value E of the voltage, such that the total energy per cycle could be represented by $E^2/r \cdot T$; in other words, E^2/r should give the average power. Since r is constant, we see that the effective value of the voltage is defined in exactly the same manner as the effective value of the current, namely, that its square is equal to the mean value of the squares of instantaneous values e. Hence, for sinusoidal voltages

$$E = E_m / \sqrt{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Power is also expressed in direct-current circuits by $e.i.$ so that it is natural to expect that in alternating-current circuits, without phase displacement, the average power should be equal

*This operation can be performed graphically in rectangular co-ordinates by plotting a curve of i^2, or in polar co-ordinates by plotting a curve of i directly.

to the product $E \cdot I$ of the effective values. This is easily verified by substituting the values of E and I from eqs. (30) and (31) into eq. (27). We have

$$W = I_m/\sqrt{2} \cdot I_m r/\sqrt{2} \cdot T = I \cdot E \cdot T \quad . \quad . \quad . \quad (32)$$

To find the average power divide both sides of this equation by T; this gives

$$P = E \cdot I \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

which was to be proved.

PROB. 41. Prove that the average power with sinusoidal currents and voltages is equal to $\frac{1}{2} I_m E_m$, or equal to $E \cdot I$, by directly integrating the expression $ei \cdot dt$ for infinitesimal energy. Use the method shown in problem 38.

The preceding discussion and examples show the advantages of using the effective values of currents and voltages, instead of the amplitudes, when drawing vectors. The chief advantage is that the average power can be calculated as in direct-current circuits, without dividing the product of the amplitudes by some factor, for instance 2 in case of the quantities varying according to the sine law. Commercial ammeters and voltmeters, intended for use on alternating-current circuits, are always calibrated to indicate the effective values of currents and voltages (See the author's Experimental Electrical Engineering, pp. 38, 44, and 47).

PROB. 42. An electric heater was tested for power consumption on an alternating-current circuit, by having an ammeter in series with it, and a voltmeter across its terminals. Both instruments were calibrated to indicate effective values. The readings were: $E = 110$ volts., $I = 5.7$ amp. Assuming the current and the voltage to have been in phase, which is nearly the case, what was the average power consumption of the heater, and what was its resistance? Determine also the maximum instantaneous values of the current and the voltage, under the supposition of the sine law. Ans. $P = 627$ watts; $r = 19.3$ ohm.; $E_m = 155.56$ volt.; $I_m = 8.06$ amp.

10. Power when Current and Voltage are out of Phase. In most practical alternating-current circuits there is a more or less pronounced phase displacement between the current and the voltage. This is due to the presence of counter-electromotive

forces in the circuit, the principal among these being: (a) The counter-electromotive forces of the motors connected into the circuit. (b) The counter-electromotive forces induced by alternating magnetic fluxes in the circuit. These fluxes may be those created by the current itself, or they may be due to the influence of other circuits (self- and mutual induction). (c) Counter-electromotive forces due to the elasticity of the dielectric medium surrounding the circuit (electrostatic capacity).

The actual workings of these causes are discussed more in detail in the following chapters and in the comparsion pamphlet on "The Magnetic Circuit." Here it is sufficient to note that there are factors producing counter-electromotive forces in alternating-current circuits, and that they bring about a phase displacement between the voltage and the current. The cause of this displacement can be understood with reference to Fig. 6. Let OB be the generator voltage, and let OA represent that part of OB which is equal and opposite to the sum of the various counter-electromotive forces in the circuit. Subtracting OA from OB the net voltage OC is obtained which is just sufficient to supply the ohmic drop in the circuit. The current OD is in phase with this voltage, and is numerically equal to OC divided by the total resistance r of the circuit. It will be seen that there is a phase displacement ϕ between the current and the generator voltage OB; it is clear from the figure that this phase displacement is due to the presence of the counter-electromotive force BC.

Consider first the specific case when the phase displacement between the current and the voltage is exactly 90 degrees. If the current is represented by eq. (18), the expression for the voltage is $e = E_m \cos x$. The instantaneous power is equal to $i \cdot e = I_m E_m \sin x \cos x = I_m E_m/2 \cdot \sin 2x$. Thus, the power varies as a sine function of double the generator frequency; the energy flows now away, and now toward the generator. The *average* power for one cycle is therefore zero, for the power has all the negative values that it has positive ones. Mathematically, this result is represented by the time integral of the instantaneous power over a complete cycle. Omitting the constant quantities E_m and I_m we have

$$\int_0^{2\pi} \sin x \cos x \, dx = \tfrac{1}{4} \Big[-\cos 2x \Big]_0^{2\pi} = 0. \quad . \quad . \quad (34)$$

Let now the phase displacement between the current and the voltage be less than 90 degrees, and be equal, say, to ϕ. The average power delivered by the alternator is in this case smaller than the product EI, and its value must be deduced separately. The voltage E_m can be resolved into a component $E_m \cos \phi$ in phase with the current, and another component, $E_m \sin \phi$, in quadrature with the current. According to the proof given above, the power produced by the second component of the voltage is zero, so that the total average power is [see eq. (28a)]:

$$P = \tfrac{1}{2} E_m \cos \phi \cdot I_m = \tfrac{1}{2} E_m I_m \cos \phi. \quad . \quad . \quad . \quad (35)$$

Replacing the amplitudes by the effective values, this equation is converted into the usual expression for the average power :

$$P = EI \cdot \cos \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

A more rigid deduction of this expression is given in problem 45 below.

The product EI is sometimes called *the apparent power*, and $\cos \phi$ is usually referred to as *the power factor*. Thus, the power factor can be defined either as the cosine of the angle of phase displacement between the current and the voltage, or as the ratio of the true power P to the apparent power IE.

Referring to Fig. 6, the factor $I \cos \phi$ which enters into eq. (36) represents the projection of I upon the direction of the voltage OB, or E. Hence, eq. (36) can be interpreted by saying that the true power is equal to the product of the voltage by the component of the current in phase with it. This component of the current is called therefore *the energy component*, while the component of the current at right angles, or in quadrature with the voltage, is called *the wattless component*. The quadrature component is wattless because the cosine of the angle between it and the voltage is zero.

Instead of resolving the vector of the current into components, it is sometimes preferable to resolve the voltage E into the components $E \cos \phi$ and $E \sin \phi$, in phase and in quadrature with the current. In this case, eq. (36) is expressed in words by saying that the average power is equal to the current times the component of the voltage in phase with it. These components of the voltage are called *the energy component* and *the wattless component* of the voltage respectively.

PROB. 43. Assuming the line current in problem 34 to be 452 effective amperes, calculate the average power P_1 delivered by the alternator, and the power P_2 received at the opposite end of the line. $P_1 = 2444$ kw.; $P_2 = 2350$ kw.

PROB. 44. Referring to problem 42, a wattmeter was connected into the heater circuit, and the true power was measured to be 598 watts. Assuming all the three instruments to be in calibration, calculate the power factor and the angle of displacement between the current and the voltage in the heater; also the energy and the wattless components of the current. Ans. $\cos\phi =$ 95.38 per cent; $\phi = 17° 30'$; 5.388 amp.; 1.714 amp.

PROB. 45. Deduce expression (35) for power in inductive circuits directly from the general expression (24a). Solution: Let the current be expressed by eq. (18), and let the voltage be leading by an angle ϕ. The expression for the voltage is then

$$e = E_m \sin(x + \phi). \quad . \quad . \quad . \quad . \quad . \qquad (37)$$

Substituting these values into eq. (24a) we get

$$\int_0^T eidt = E_m I_m \int_0^T \sin x \sin(x + \phi)\, dt$$

$$= E_m I_m T/2\pi \cdot \int_0^{2\pi} \sin x \Big[\sin x \cos\phi + \cos x \sin\phi\Big] dx$$

$$= E_m I_m T/2\pi \cdot \Big[\cos\phi \int_0^{2\pi} \sin^2 x\, dx$$

$$+ \sin\phi \int_0^{2\pi} \sin x \cos x\, dx\Big].$$

The value of the first integral is π [see eq. (26)]; the second integral is equal to zero, according to eq. (34). Substituting these values and dividing both sides of the equation by T, formula (35) is obtained.

PROB. 46. Draw sine waves of the current and of the voltage in problem 27; also plot the curve of instantaneous values of power. Check the average ordinate of the power curve with formula (35). Explain how power is negative during a part of the cycle, remembering that there are counter-electromotive forces in the circuit.

PROB. 47. Prove that the curve of power consists of a sine wave of double frequency, plus a constant term, the latter representing the average power. Suggestion: Compare with problem 37, $ie = I_m E_m \sin x \sin (x + \phi)$. Use the trigonometric transformation: $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$.

PROB. 48. Determine the ratio of the average ordinate of a sine curve to its amplitude. Solution: The average ordinate for one quarter of the cycle is determined from the condition

$$Y_{ave} \cdot \pi/2 = \int_0^{\pi/2} Y_m \cdot \sin x \, dx = Y_m$$

whence

$$Y_{ave}/Y_m = 2/\pi = \cdot 6366. \quad . \quad . \quad . \quad . \quad (38)$$

PROB. 49. The ratio of the effective value of an alternating current or voltage to its average value is called *the form factor*. Determine the values of the form factor for the rectangular wave, the sine wave, and the triangular wave. Ans. 1.000; 1.111; 1.155.

CHAPTER IV.

REACTANCE AND RESISTANCE IN ALTERNATING-CURRENT CIRCUITS.

11. **Inductance and Reactance.** This chapter is intended to supplement Chapter V of the author's "Experimental Electrical Engineering." The student is referred to that book for the physical concepts and definitions of inductance and reactance, and for the fundamental experiments which illustrate the properties of conductors in this respect. As a matter of fact, it is not necessary for the understanding of this chapter to know that the cause of the counter-electromotive force of inductance is the magnetic flux. It is sufficient to accept as an experimental fact that electricity in motion possesses some kind of inertia. This inertia is analogous to that of a fluid in a pipe. When the motion of the fluid is retarded or accelerated by an external force, the inertia brings into play a reactive force which tends to oppose the change in velocity.

In the case of an alternating current, which varies according to the sine law, this reaction or counter-electromotive force also varies according to the sine law. In practice, it is often more convenient to consider the voltage applied at the terminals of the reactance, equal and opposite to the above mentioned counter-electromotive force. This voltage is expressed by equation (3) on p. 118 of the "Experimental Electrical Engineering." This equation, as well as eq. (4) containing the definition of reactance, form the basis of all the further discussion; it is therefore essential that the student should make their significance clear to himself, before proceeding further. The fundamental relations are also explained in Figs. 102, 108, and 109. The results reached are briefly as follows:

(1) When an alternating current is flowing through a conductor, the alternating flux created around the conductor induces in it a counter-electromotive force. This electromotive force lags in phase by 90 degrees with respect to the current, and is numerically equal to $2\pi fLI$. Here I is the effective value of the current, in amperes; the voltage is expressed in effective volts.

L is a coefficient which characterizes the conductor with respect to its ability to create counter-electromotive force, with changing current.* In this chapter, the inductance L is supposed to be given or measured experimentally.

(2) The voltage applied at the terminals of the inductance is equal and opposite to the counter-electromotive force, and consequently leads the current by 90 degrees.

(3) The quantity $2\pi fL$ which enters into voltage calculations is called the *reactance* of a conductor or a winding. The reactance multiplied by the current I gives the voltage drop in the conductor; therefore, the reactance is measured in ohms, like resistance.

(4) When a conductor possesses both resistance and reactance, or when a resistance and a reactance are connected in series, the total voltage drop is represented by the hypothenuse of a right triangle, the two other sides representing the ohmic and the inductive drops respectively. Consequently, resistance and reactance must be added at right angles, as is shown in Fig. 108; their combined effect is called the *impedance* of the conductor.

(5) Total voltage drop across a conductor, which possesses both resistance and reactance, is equal to the impedance of the conductor multiplied by the current. Referring to Fig. 109, it will be seen that the power factor, cos ϕ, is equal to $ir/iz = r/z$. In other words, the ratio of the resistance to the impedance gives the power factor, cos ϕ, and thus determines the angle ϕ by which the current lags behind the voltage.

PROB. 50. The inductance of a coil is .2 henry, its ohmic resistance is negligible. Draw a curve giving the voltage necessary to maintain a current of 12 amps. through the coil, at frequencies ranging from zero to 100 cycles per second. Ans. A straight line through the origin; at $f = 100$, $E = 1508$ volts.

PROB. 51. A reactive coil without iron draws a current of 75 amps. when connected across a 110 volt, 25 cycle circuit. What current would it draw at 60 cycles and at the same voltage, provided that the effect of its resistance can be neglected? Plot a curve of current at intermediate frequencies. Ans. 31.25 amps.; equilateral hyperbola asymptotic to both axis.

* The physical meaning of L in terms of magnetic lines of force, and its calculation are treated in the pamphlet on "The Magnetic Circuit."

PROB. 52. A 2200 v., 600 kw, 50 cycle transformer must have a wattless magnetizing current of not over 2.5 per cent of the full-load current. What is the lower limit of its no-load reactance and inductance? Ans. 322.5 ohm; 1.027 henry.

12. Problems on Impedance, and Impedances in Series. The meaning of the impedance, and the numerical relations between it, the voltage, and the current are explained in Figs. 102, 108, and 109 of the Experimental Electrical Engineering. These figures also show the angle ϕ of displacement between the voltage and the current. The relations (10) and (11) on p. 125 are also useful in the solution of problems.

PROB. 53. The coil considered in problem 50 is connected in series with a 100 ohm resistance; it is required to maintain a current of 12 amperes through the two, at various frequencies. Supplement the curve obtained in problem 50 with curves of voltage drop across the resistance, and the total voltage across the combination. Plot also the corresponding values of power factor. Determine the ordinates of the curves graphically, and check a few analytically. Ans. $E_r = 1200$ v. independent of the frequency. At $f = 0$, $E_z = E_r$, and $\cos\phi = 1$. At $f = 100$, $E_z = 1927$ v., $\cos\phi = 62.25$ per cent. The curves resemble those in Fig. 104.

PROB. 54. What is the value of impedance in the preceding problem, at $f = 100$? Ans. 160.6 ohm.

PROB. 55. When a certain non-inductive resistance is connected across a source of alternating voltage, a current i flows through it. When an inductance, containing negligible resistance, is connected across the same source of voltage, the current is i'. What is the current and the phase displacement when the resistance and the inductance are connected in series across the same source? Solution: Let the unknown voltage be E. The unknown resistance is $r = E/i$; the unknown reactance $x = E/i'$. When the two are connected in series, the impedance

$$z = \left[(E/i)^2 + (E/i')^2\right]^{1/2}.$$

Consequently, the current is

$$E/z = ii'/(i^2 + i'^2)^{1/2}\,;\ \tan\phi = x/r = i/i'.$$

PROB. 56. Supplement the curves in problem 53 by those of true and apparent power. Ans. 14.4 kw. at all frequencies; 23.12 kv-amp. at $f = 100$.

PROB. 57. The instantaneous readings on the instruments in a power house are: 7520 kw., 66 kv.; 147 amps. The power-factor meter shows that the current is lagging behind the voltage. What are the readings at the same instant at the receiving end of the line, if its resistance is 45 ohms, and its reactance 83 ohms. Ans. 6547 kw.; 53.4 kv. Hint: Draw the vectors of the generator voltage and current in their true relative position. Subtract the ohmic drop in phase with the current, and the inductive drop in quadrature with it. The result gives the receiver voltage in its true magnitude and phase position.

PROB. 58. Two impedance coils are connected in series across 292 alternating volts. The voltages across the coils are 152 v. and 175 v.; the current is 7.3 amps. Knowing that the resistance of the first coil is 10 ohms, determine graphically, as in Fig. 114, the resistance of the second coil; also the reactances and the impedances of both coils. Ans. $r_2 = 12.95$; $z_1 = 20.82$; $z_2 = 23.97$, all in ohms.

PROB. 59. In order to determine the power input into a single-phase 110 v. motor, without the use of wattmeter, the motor is connected in series with a non-inductive resistance across a 220 v. circuit. The resistance is adjusted so that the voltage across the motor terminals is 110 v. when the motor is carrying the required load. Under these conditions the voltage across the resistance is found to be 127 v., and the current through the motor 23 amps. From these data determine graphically the power factor of the motor, and calculate the power input into it, as is explained in Sec. 109 of the Experimental Electrical Engineering. Ans. 72.27 per cent; 1826 watts.

PROB. 60. Referring to the preceding problem, calculate $\cos \phi$ trigonometrically, from the triangle of voltages, instead of determining it graphically.

13. Impedances in Parallel. The problems that follow are intended to illustrate the theory given in Sections 111–113 of the Experimental Electrical Engineering.

PROB. 61. Let the resistance and the inductance mentioned in problem 55 above be connected in parallel. What is the total

current, and the phase displacement between the current and the voltage? Ans. $I = (i^2 + i'^2)^{1/2}$; $\tan\phi = i'/i$.

PROB. 62. The load of a single-phase, 6600 v. generator is estimated to consist of 1200 kw. of lamps, practically non-inductive, and of 800 kw. of motors, working at an average power factor of 75 per cent. What will be the expected generator output, in amperes, and the power factor? Solution: The energy component of the motor current is $800/6.6 = 121.2$ amp.; the wattless component is $121.2 \tan\phi = 106.8$ amp. The lamp current is $1200/6.6 = 181.8$ amp. The total energy component of the generator current is $121.2 + 181.8 = 303$ amp. Consequently, total generator current is $(303^2 + 106.8^2)^{1/2} = 321.3$ amp.; the power factor is $303/321.3 = 94.3$ per cent.

PROB. 63. Check the solution of the preceding problem graphically, as in Fig. 121.

PROB. 64. Let the load given in prob. 62 be distributed equally among the phases of a three-phase system. What is the total current per phase? Ans. 185.5 amp.

PROB. 65. Three impedance coils, having ohmic resistances of 2, 3, and 4 ohms. respectively, and inductances of 13, 10, and 22 millihenrys, are connected in parallel across a source of 220 v., 60 cycle alternating voltage. Construct the vector of the resultant current, as in Fig. 119. Ans. 110 amp.; $\cos\phi = .495$.

PROB. 66. Solve the preceding problem for the frequency of 25 cycles per second. Construct both diagrams to the same scale so as to see the influence of frequency.

PROB. 67. In problem 65 let the total current be given in magnitude, but not in its phase position; assume the inductance of the third coil to be unknown. Show how to determine graphically the vector of the current in the third coil, and the position of the vector of the total current.

14. Equivalent Resistance and Reactance. Let a resistance r and a reactance x be connected *in series* across a source of alternating voltage E. Let another resistance r_p and a reactance x_p be connected *in parallel* across the same source. The latter resistance and reactance can be selected so as to give the same total current and the same phase displacement as the first

combination. The two combinations can be called *equivalent*, and used one instead of the other in all calculations involving the relations between currents and voltages. This provides a method of reducing complicated circuits to simpler circuits.

Let it, for instance, be required to add two impedances in parallel. Each impedance can be replaced by an equivalent combination of a resistance and a reactance in parallel; then the two resistances and the two reactances can be replaced by one resistance and one reactance respectively. Finally, this parallel combination can be replaced by an equivalent combination of a resistance and a reactance in series. The details of calculations are explained in problem 76 below.

The problem is: knowing r and x to find r_p and x_p. The conditions for the two combinations to be equivalent are expressed analytically as follows: The current through r_p is E/r_p, in phase with E; the current through x_p is E/x_p, in quadrature with E. Consequently, the total current through the equivalent combination is $(E^2/r_p^2 + E^2/x_p^2)^{1/2}$. This must be equal to the current E/z through the original combination of r and x in series. Hence, equating the two, we have

$$1/r_p^2 + 1/x_p^2 = 1/z^2. \quad \text{. . . . . .} \quad (39)$$

In this equation z is a known quantity, being equal to $(r^2 + x^2)^{1/2}$. The angle of displacement between the total current and the voltage in the equivalent combination is determined from the relation: $\tan \phi = (E/x_p)/(E/r_p) = r_p/x_p$. For the given combination in series we have: $\tan \phi = x/r$. Since, by assumption, both combinations are to produce the same phase displacement, we have

$$r_p/x_p = x/r. \quad \text{. . . . .} \quad (40)$$

Solving eqs. (39) and (40) for the unknown quantities r_p and x_p, and remembering that $r^2 + x^2 = z^2$, we obtain

$$\left.\begin{aligned} r_p &= z^2/r \\ x_p &= z^2/x \end{aligned}\right\} \quad \text{. . . . .} \quad (41)$$

PROB. 68. An impedance coil having a resistance equal to 2 ohm. and a reactance of 7.5 ohm. are to be replaced by an equivalent resistance and reactance in parallel. Find their values. Ans. $r_p = 30.125$ ohm.; $x_p = 8.035$ ohm.

PROB. 69. Check the solution of the preceding problem by actually calculating the current and the phase angle for the two combinations, at a certain assumed voltage.

PROB. 70. Show that r_p and x_p are always larger than r and x respectively. Hint: Replace z^2 in eqs. (41) by $r^2 + x^2$.

PROB. 71. Let r_p and x_p be given; find the values of r and x. Solution: Solve eqs. (41) for r and x, using the value of z^2 calculated from eq. (39).

PROB. 72. In adjusting a measuring instrument a non-inductive resistance of 120 ohm. was used in parallel with a choke coil. The impedance of the coil was 75 ohm. its resistance 16 ohm. In the regular manufacture of the instrument it is desired to use a resistance and a reactance in series. Determine their values, either graphically or analytically. Ans. $r = 37.9$ ohm.; $x = 44.15$ ohm.

15. Admittance and its Components—Conductance and Susceptance. It is shown in Sec. 2 that when resistances are connected in parallel it is more convenient to use their reciprocals—conductances. It is natural to expect that the same would apply to the addition of reactances and of impedances. This leads to two new concepts, those of *susceptance*, a reciprocal of reactance, and of *admittance*, a reciprocal of impedance. Since reactances and impedances are measured in ohms, susceptances and admittances are measured in mhos, like conductances.

We thus have the following six quantities:

resistance	conductance
reactance	susceptance
impedance	admittance

The first three show *how difficult* it is to force a unit current through a conductor; the last three are the reciprocals, and show *how easy* it is to force a unit current through the conductor. The first three are measured in ohms, the latter three in mhos. The first three form a right-angle triangle, it is shown below that the latter three also form a similar triangle.

If the resistance of a conductor is 5 ohms, its conductance is 0.2 mho. In the same way, if the reactance of a coil is equal to 5 ohms, its susceptance is said to be 0.2 mho (provided, that its ohmic resistance is negligible). It is necessary to distinguish

between the conductance and the susceptance, because in one case the current is in phase with the voltage, in the other case it is in quadrature, lagging behind the voltage. If the impedance of a coil is 5 ohm., it being understood that the coil has both ohmic resistance and reactance, the admittance of the coil is 0.2 mho. *This admittance cannot, however, be separated into the conductance and the susceptance, without an additional definition of the latter*, even though the resistance and the reactance of the coil be known.

It is customary to define the conductance and the susceptance of an impedance coil as those of its equivalent parallel combination. The reason for this is that conductances and susceptances are used when impedances are added in parallel. As is explained above, each impedance is in this case replaced by an equivalent resistance r_p and reactance x_p in parallel. The reciprocals of these are called the conductance and the susceptance of the original coil; it must be remembered, however that in reality they refer to the equivalent parallel combination.

Let us assume, for instance, that the above mentioned impedance coil possesses a resistance of 3 ohm, and a reactance of 4 ohm. According to eqs. (41), the equivalent resistance is 25/3 ohm., the equivalent reactance is 25/4 ohm. Therefore, by definition, the conductance of the original coil is $3/25 = .12$ mho., and its susceptance is $4/25 = .16$ mho. It will be seen that these values are different from the erroneous values of 1/3 and 1/4 obtained by simply taking the reciprocals of the resistance and the reactance of the coil itself.

PROB. 72*a*. An apparatus takes 25 amp. and 2000 watts at 110 volt., and the current is lagging. What is the equivalent conductance and susceptance of the device? What is the resistance and reactance in series equivalent to this apparatus? Ans. .165 mho.; .156 mho.; 3.2 ohm.; 3.02 ohm.

Susceptances are added in parallel like conductances, as may be seen from the following proof, similar to the proof of eq. (7). Let several coils possessing reactances x_1, x_2, etc., and no ohmic resistance whatever, be connected in parallel across a source of alternating voltage E. The currents through these coils are: $i_1 = E/x_1$; $i_2 = E/x_2$, etc. All these currents are in phase with one another, being displaced by 90 degrees with respect to the

voltage E. An equivalent reactance x_{eq} is by definition such as would give the same total current as the combination of the given reactances in parallel. We have therefore

$$E/x_{eq} = \Sigma E/x, \text{ or } 1/x_{eq} = \Sigma 1/x.$$

Denoting the reciprocals of the reactances, or the susceptances of the coils, by b, we have

$$b_{eq} = b_1 + b_2 + \text{etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

This is analogous to eq. (7). We also have for each branch

$$I = bE. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

This corresponds to eq. (6), except that the current is understood to be lagging behind the voltage by 90 degrees.

PROB. 73. Three resistances of 2, 5, and 10 ohm., and two reactances of 4 and 2.5 ohm. are all connected in parallel across 250 alternating volts. What is the total current and the combined power factor? Solution: The equivalent conductance is $1/2 + 1/5 + 1/10 = .8$ mho. The equivalent susceptance is $1/4 + 1/2.5 = .65$ mho. The energy component of the current, according to eq. (6), is $250 \times .8 = 200$ amp. The wattless component, according to eq. (43), is $250 \times .65 = 162.5$ amp. Hence, $\tan \phi = 162.5/200 = .8125$; $\cos \phi = .775$; total current $= 200/.775 = 258$ amp.

PROB. 74. Assuming the admittance of a coil to be defined as the reciprocal of its impedance, express this admittance through the conductance and the susceptance of the coil. Ans. Eq. (39) gives directly

$$y^2 = g^2 + b^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

where $y = 1/z$ is the admittance.

PROB. 75. Express the conductance and the susceptance of a coil through its resistance and reactance. Ans. Eqs. (41) gives

$$g = r/z^2 \qquad b = x/z^2 \quad . \quad . \quad . \quad . \quad . \quad (45)$$

or, using the admittance $y = 1/z$,

$$g = ry^2 \qquad b = xy^2. \quad . \quad . \quad . \quad . \quad . \quad (46)$$

PROB. 76. Two impedances z_1 and z_2, consisting respectively of resistances r_1 and r_2, and of reactances x_1 and x_2, are connected in parallel. Find the resistance and the reactance of the equiva-

lent impedance coil. Solution: Replacing each coil by a resistance and a reactance in parallel, and adding their conductances and susceptances, we have: $g_{eq} = r_1/z_1^2 + r_2^2/z_2^2$; similarly $b_{eq} = x_1/z_1^2 + x_2/z_2^2$. The admittance of the equivalent coil is, according to eq. (44), $y_{eq} = (g^2_{eq} + b^2_{eq})^{1/2}$. Eqs. (46) give then: $r_{eq} = g_{eq}/y^2_{eq}$; $x_{eq} = b_{eq}/y^2_{eq}$.

PROB. 77. Extend the solution of the preceding problem to the case of more than two impedances in parallel. Ans. $g_{eq} = \Sigma r/z^2$; $b_{eq} = \Sigma x/z^2$. The rest of the solution is the same.

PROB. 78. The admittance of a winding is .2 mho; the current through the winding lags by 34 degrees with respect to the voltage at its terminals. Determine the resistance and the reactance of the winding. Ans. $r = 4.145$ ohm.; $x = 2.796$ ohm.

PROB. 79. A coil, having a resistance of a 2.3 ohm. and a reactance of 5 ohm, is connected in parallel with another coil, for which $r = 3$ ohm. and $x = 4$ ohm. Calculate the resistance and the reactance of the equivalent coil. Ans. $r = 1.36$ ohm.; $x = 2.255$ ohm.

PROB. 80. The coils given in problem 79 are connected in parallel across 55 volts. Calculate the total current, its energy and wattless components, and the power factor of the combination. Ans. 20.85 amp.; 10.776 amp.; 17.88 amp.; $\cos\phi = .5165$.

PROB. 81. Prove that the ratio of r/x is equal to the ratio of g/b; in other words, that the impedance triangle is similar to the admittance triangle.

PROB. 82. Check the value of the total current in problem 65 by calculating the combined admittance of the three coils.

CHAPTER V.

THE USE OF COMPLEX QUANTITIES.

16. Addition and Subtraction of Vectors in Projections. With the explanation given in the preceding four chapters the student is enabled to handle, by means of vector diagrams, problems involving resistances and reactances in alternating-current circuits. Several problems in transmission-line calculations and in the theory of alternating-current machinery are reduced to such vector diagrams of electric circuits. The disadvantages of the graphical method are: (1) Results are usually obtained which hold for one specific case only; an analysis of the effect of various factors is often difficult. (2) Some vectors may be many times smaller than others, for instance, the voltage drop in a line, as compared to the line voltage itself. Therefore, the diagram must be drawn to a very large scale, or else the results are not accurate enough. In addition to these drawbacks, it may be said that some people object to graphical methods in general, as involving the use of drawing instruments which may not be handy.

On the other hand, vector diagrams are quite convenient in some practical cases; moreover, they are helpful for the understanding of general relations in a circuit, without reference to particular numerical values. Again, in some problems, the unknown vectors can be calculated from the vector diagram trigonometrically, without the necessity of actually drawing the diagram to scale.

It is possible to treat vectors analytically, using their projections on two axes, as in analytic geometry (Fig. 7). A vector, such as E, can be defined either by its magnitude and phase angle ϕ, or it can be given by its projections e and e' upon the axes of coordinates. If E and ϕ are given, the projections are calculated from the expressions $e = E \cos \phi$; $e' = E \sin \phi$. If the

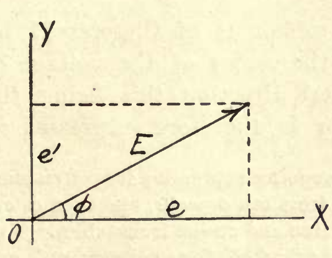

FIG. 7. A vector and its projections.

projections are given, the vector itself is determined by $E = (e^2 + e'^2)^{1/2}$, and $\tan \phi = e'/e$.*

The fact that e and e' are components of the vector E along two perpendicular axes is expressed symbolically thus :

$$\underset{\cdot}{E} = e + je'. \qquad (47)$$

Here j is a symbol which indicates that the projection e' refers to the vertical axis. This symbol must not have any *real* value ; for the time being, it may be considered merely as an abbreviation of the words : "along the vertical axis." The sign plus in eq. (47) denotes the geometric addition. The dot under E signifies that by $\underset{\cdot}{E}$ is meant not only the magnitude of the vector, but its direction as well, the latter being defined by the projections. When the magnitude only is meant, the dot is omitted.

The foregoing notation has been introduced by Dr. Chas. P. Steinmetz, and is now universally used in this country, in the treatment of alternating currents. Much credit is also due to Dr. Steinmetz for developing the analytic method of dealing with alternating currents and voltages by means of their projections.

Addition and substraction of vectors in projections is reduced simply to the addition and subtraction of the projections : According to Fig. 5, the projection of an equivalent vector on any axis is equal to the sum of the corresponding projections of the component vectors on the same axis. Thus, if a current is represented as a vector by its projections $50 + j70$ amp., and another current by $100 + j40$ amp., the vector sum of these currents is $150 + j110$ amp. Or, the resultant of two voltages, $\underset{\cdot}{E}_1 = e_1 + je_1'$ and $\underset{\cdot}{E}_2 = e_2 + je_2'$ is

$$\underset{\cdot}{E}_{eq} = \underset{\cdot}{E}_1 + \underset{\cdot}{E}_2 = (e_1 + e_2) + j\,(e_1' + e_2').$$

As an illustration, let us solve problem 33 of Chapter II by the method of projections. Take the vector of the voltage of the first alternator in the horizontal direction, this being the simplest assumption. This vector is therefore expressed as

* Instead of determining E from the foregoing expression it is often more convenient first to determine the angle ϕ from $\tan \phi = e'/e$, and then calculate $E = e/\cos \phi$, or $E = e'/\sin \phi$, taking sine and cosine from tables. This method is used in the solution of problem 73, where the total current is calculated from its components.

$\dot{E}_1 = 2300 + j0$. The horizontal projection of the second vector is $1800 \cos 27° = 1603.8$ v., and its vertical projection is $1800 \sin 27° = 817.2$ v. Both of these projections are positive, because the second vector leads the first, and is therefore in the first quadrant. Thus, $\dot{E}_2 = 1603.8 + j817.2$ v. The resultant voltage, $\dot{E}_{eq} = \dot{E}_1 + \dot{E}_2 = 3903.8 + j817.2$ volts. For some purposes, it is sufficient to leave the answer in this form, that is to say, to give only the projections of the resultant voltage. If, however, the magnitude and the phase position are required, they can be found as explained above: $\tan \phi = 817.2/3903.8 = .2092$; $\phi = 11° 49'$; $\cos \phi = .9786$; $E_{eq} = 3903.8/.9786 = 3988$ volt.

If, however, the terminals of the second machine be reversed, then, $\dot{E}_{eq} = \dot{E}_1 - \dot{E}_2 = 696.2 - j817.2$. This vector has a positive horizontal projection and a negative vertical projection. Consequently, the vector lies in the fourth quadrant, and lags behind the reference vector by less than 90°. Proceding as above, we find: $\phi = -49° 32'$, $E_{eq} = 1074$ volts.

PROB. 83. Solve problem 34 by the method of projections, assuming the vector of the voltage to be horizontal.

PROB. 84. Check the solution of problem 30 by the method of projections.

PROB. 85. Verify the answer to problem 65 by the method of projections.

17. Rotation of Vectors by Ninety Degrees. In problems involving reactance it is necessary to turn the vector of the current by 90° and then multiply it by the reactance in order to determine the reactive drop in voltage. Multiplication of the vector of the current by the reactance converts it into a vector of voltage, and thus merely changes its scale. But turning the vector modifies the relative magnitudes of its projections; it is, therefore, necessary to find a relation between the original and the new magnitudes of the projections.

Let in the simplest case a vector $\dot{E}_1$ be drawn along the reference axis of abscissae, and let its length be a. In the symbolic notation it is represented as $\dot{E}_1 = a$, the other projection being zero. After having been turned by 90° counter clock-wise, the vector is directed along the positive axis of ordinates, and can

be symbolically represented as $E_2 = ja$, the horizontal projection being zero. Thus, in this particular case, a rotation by 90 degrees is equivalent to multiplication by j.

It is convenient to define j so that multiplication of a vector by j will mean a rotation by 90 degrees in the positive direction (counter clock-wise), *while division by j will turn the vector by 90 degrees in the negative direction* (clock-wise). In order to find a value of j which satisfies these requirements, let the vector E_2 be turned again by 90° counter clock-wise, being now directed along the negative X-axis. Its expression is now $E_3 = -a$. On the other hand, the same expression must be obtained by multiplying E_2 by j. Therefore, we have $-a = j^2 a$, or $j^2 = -1$; consequently, $j = \sqrt{-1}$. If the original vector E_1 is to be turned by 90° clock-wise, we must, according to our assumption, divide it by j. We have: $E_4 = a/j$, or, multiplying the numerator and the denominator by j, $E_4 = ja/j^2 = -ja$. This checks with the preceding results, because $E_2 = -E_4$. It will thus be seen that the value of $j^2 = -1$ satisfies the requirements set above, when the original vector is directed along one of the axes of co-ordinates.

Let now the original vector E_1 (Fig 8) have an arbitrary direction in the first quadrant, or $E_1 = a + jb$. Multiplying E by j we should get the vector E_2 of the same magnitude, but in the second quadrant, and perpendicular to E_1. E_2 has a vertical projection equal to the horizontal projection a of the original vector E_1; the horizontal projection of E_2 is negative, and is equal in its absolute value to the vertical projection b of the vector E_1. Thus, the new vector is expressed as $E_2 = -b + ja$. On the other hand, multiplying E_1 by j we have $jE_1 = ja + j^2 b = ja - b$,

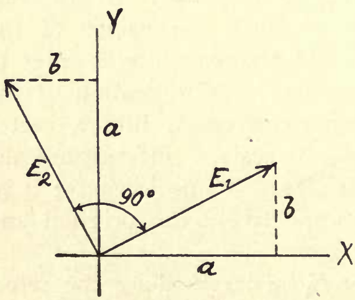

FIG. 8. Relation between the projections of two vectors, perpendicular to each other.

which is the same as above. Therefore, in this case also the assumption $j^2 = -1$ is correct and leads to rotation by 90 degrees. It is left to the student to verify the cases when the vector lies in some other quadrant, and when E_1 is divided by j, instead of being multiplied by j.

PROB. 86. A current of $80 + j43$ amps. flows through a resistance of 2 ohm. in series with a reactance of 3 ohm. Find the complex* expression for the voltage across the impedance. Solution: The vector of the voltage consists of two components, representing the ohmic and the reactive drop respectively. The ohmic drop, E_1, is equal to $2\ (80 + j43) = 160 + j86$ volt. To find the inductive drop, E_2, the vector of the current must be turned by 90 degrees, in other words, multiplied by j, and then multiplied by $x = 3$. Thus, $E_2 = 3j\,(80 + j43) = -129 + j240$ volt. Total voltage $E = E_1 + E_2 = 31 + j326$ volts.

PROB. 87. Solve the preceding problem when the voltage is given and the current unknown. *First Solution:* Let the unknown current be represented by its projections as $i + ji'$. We have, as in the preceding problem,

$$2\,(i + ji') + 3j\,(i + ji') = 31 + j326,$$

or, separating the terms with j,

$$(2i - 3i') + j\,(2i' + 3i) = 31 + j326. \quad . \quad . \quad (48)$$

This equation can be satisfied only if the terms with and without j are equal to each other separately, because a real quantity cannot be increased or diminished in its magnitude by the addition of an imaginary quantity. Or, from a geometric point of view, the left hand side and the right hand side of eq. (48) represent a vector in its projections. But two vectors are identical only when their corresponding projections are equal. Thus, we have

$$2i - 3i' = 31 : \quad 2i' + 3i = 326.$$

Solving these equations for i and i' we find $i = 80$, $i' = 43$, as is given in the preceding problem. *Second Solution:* The equation

*Expressions of the form $a + jb$, where j is $\sqrt{-1}$, are called in algebra *complex quantities*.

preceding (48) can be written in the form $(2 + 3j)(i + ji') = 31 + j326$; or, $i + ji' = (31 + j326)/(2 + 3j)$. Considering here j as an ordinary algebraic quantity, we can get rid of it in the denominator by multiplying the numerator and the denominator by $2 - 3j$. The result is

$$i + ji' = \frac{(31 + j326)(2 - 3j)}{2^2 - (3j)^2} = \frac{1040}{13} + j\frac{559}{13},$$

or, $i + ji' = 80 + j43$, as before.

PROB. 87a. A voltage of $28 + j120$ volts applied to the terminals of a coil produces in it a current equal to $4 + j1.5$ amps. Determine the resistance and the reactance of the coil. Ans. $r = 16$ ohm.; $x = 24$ ohm.

18. Impedance and Admittance expressed as Complex Quantities. The method employed in the solution of the two preceding problems can now be generalized. Let it be required to find the voltage necessary to maintain a current $i + ji'$ through a resistance r and reactance x in series. The voltage necessary for overcoming the resistance is equal to $r(i + ji')$; that for overcoming the counter-e.m.f. of the reactance is $jx(i + ji')$. Hence, the total voltage is $\dot{E} = r(i + ji') + jx(i + ji')$, or

$$\dot{E} = e + je' = (r + jx)(i + ji'). \quad . \quad . \quad . \quad (49)$$

It is legitimate to factor out the expression $(i + ji')$, and to treat j as any other algebraic quantity, because j is now assigned a definite value $\sqrt{-1}$. Moreover, eq. (49), as well as the equation, (48), represents simply the geometric addition of four component vectors, two of them being along the X-axis and the other two along the Y-axis. As long as this interpretation is kept in mind, the terms may be arranged in any desired order.

Eq. (49) shows that, in order to obtain the expression of the voltage corresponding to a current $(i + ji')$ through an impedance $z = (r^2 + x^2)^{1/2}$, the current must be multiplied by the complex expression $r + jx$. It must be kept in mind, that $r + jx$ is not a vector, but merely an *operator* upon the vector of the current. The operation consists first in multiplying the vector of the current by r, then in multiplying the same vector by x and turning it by 90 degrees in the positive direction, and finally,

in adding the two vectors geometrically. All these operations are included in the expression $r + jx$.

In order to get the projections e and e' of $\dot{E}$ from eq. (49) the terms on the right-hand side must be actually multiplied and the results represented in the form of a complex quantity. We get then

$$\dot{E} = e + je' = (ri - xi') + j(ri' + xi).$$

If the voltage and the impedance are given, and it is required to find the current, we get from eq. (49)

$$i + ji' = (e + je')/(r + jx).$$

In order to reduce the right hand side of this equation to the form of a complex quantity we multiply the numerator and the denominator by the expression $r - jx$. This gives

$$i + ji' = \frac{(e + je')(r - jx)}{r^2 - (jx)^2} = \frac{re + xe'}{r^2 + x^2} + j\frac{re' - xe}{r^2 + x^2}. \quad (50)$$

Equating the real and the imaginary parts gives the required projections of the current $\dot{I}$.

Instead of *dividing* the voltage by the operator $(r + jx)$ and then eliminating j from the denominator, it is more convenient to introduce another operator by which the voltage must be *multiplied* in order to obtain the current. It will be seen from eq. (50) that such an operator is

$$\frac{1}{r + jx} = \frac{r - jx}{r^2 + x^2} = \frac{r}{r^2 + x^2} - j\frac{x}{r^2 + x^2}.$$

Remembering that $r^2 + x^2 = z^2$, and with reference to eqs. (45), we see that the foregoing operator is simplified to $(g - jb)$, where g is the conductance and b the susceptance of the circuit. We have thus, analogously to eq. (49),

$$\dot{I} = i + ji' = (g - jb)(e + je'). \quad . \quad . \quad . \quad (51)$$

The expression $g - jb$ may be properly called the *inverse* operator, because it is equal to $1/(r + jx)$. The inverse operator is convenient to use when the voltage is given, while the direct operator is employed when the current is known (see also problem 98 below).

Equation (49) can be said to express that the voltage $\underset{\cdot}{E}$ is equal to the product of the current by the impedance, if the operator $(r + jx)$ be considered as the impedance of the circuit in the complex notation. Denote the impedance by Z, then

$$Z = r + jx. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (52)$$

Here capital Z is used to indicate that it is a complex quantity, as distinguished from the numerical value z of the same impedance. The letter is not provided with a dot, because Z is not a vector, but an *operator*.

In an abbreviated notation, eq. (49) can be written as

$$\underset{\cdot}{E} = \underset{\cdot}{I}Z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (53)$$

It must be understood that in this expression each letter stands for a complex quantity, so that when actual numerical or algebraic relations are necessary, the expression must again be expanded into (49) and the multiplication of the two complex quantities actually performed.

Similarly, eq. (51) can be written in the abbreviated form as

$$\underset{\cdot}{I} = \underset{\cdot}{E}Y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (54)$$

where Y is the inverse operator, or the admittance of the circuit. In the complex notation

$$Y = g - jb. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (55)$$

Compare eqs. (53) and (54) with eqs. (1) and (6) of Chapter I. It will be seen that the former represent the generalized Ohm's law in application to alternating-current circuits. Keeping the meaning of the symbols in mind, it is possible to solve alternating-current problems in the symbolic notation almost as easily as direct-current problems. The following problems are intended to illustrate the method.

PROB. 88. Find, by means of complex quantities, the voltage required in problem 57. Solution: Assume the vector of the current to be the reference vector. At the power house $\cos\phi = 7520/66 \times 147 = .775$; $\sin\phi = .632$; The generator voltage $\underset{\cdot}{E}_g = 66 \times .775 + j66 \times .632 = 51.15 + j41.71$ kilovolt. The voltage drop in the line is $147\,(45 + j83) = 6615 + j1220$ volt.

Hence, the load voltage $\dot{E} = (51.15 - 6.61) + j(41.71 - 12.2) = 44.54 + j29.51$ kilovolt. The numerical value of the load voltage $E = (44.54^2 + 29.51^2)^{1/2} = 53.43$ kilovolt.

PROB. 89. Determine analytically the resistence r_2 required in problem 58.

PROB. 90. A voltage of $180 + j75$ volt. produces a current of $7 + j1.5$ amp. What is the impedance of the circuit? Ans. $26.78 + j4.97$ ohm.

PROB. 91. Power is transmitted from a single-phase alternator to a load consisting of a resistance of 1.17 ohm in series with a reactance of .67 ohm. The generator voltage is 2300 v., and the impedance of the transmission line $= .085 + j.013$ ohm. Determine (a) the line current; (b) the voltage drop in the line; (c) the receiver voltage. Take the generator voltage as the reference vector. Ans. (a) $1413.6 - j769.6$ amp.; (b) $130.1 - j47$ volt.; (c) $2169.9 + j47$ volt. Use the inverse operator to obtain the current, and the direct operator to calculate the line drop.

PROB. 92. A voltage, $e + je'$, is impressed across the impedances $r_1 + jx_1$ and $r_2 + jx_2$ in parallel. Find the total current. Solution: The total conductance is $g = r_1/z_1^2 + r_2/z_2^2$, and the total susceptance is $b = x_1/z_1^2 + x_2/z_2^2$. Hence, the current $i + ji' = (e + je')(g - jb) = (eg + e'b) + j(e'g - eb)$.

PROB. 93. Extend the solution of problem 92 to the case where more than two impedances are in parallel. Ans. $i + ji' = [e \Sigma r/z^2 + e' \Sigma x/z^2] + j[e' \Sigma r/z^2 - e \Sigma x/z^2]$.

PROB. 94. Two impedances, $r_1 + jx_1$ and $r_2 + jx_2$, in parallel, are connected in series with a third impedance $r + jx$. Show how to determine the total voltage, knowing the total current $i + ji'$; or, how to find the expression for the total current when the total voltage $e + je'$ is given.

PROB. 95. Show how to solve the preceding problem when both the current and the voltage are given, but either the impedance $r_1 + jx_1$ or the impedance $r + jx$ is unknown.

PROB. 96. Four given resistances and reactances are connected as shown in Fig. 9, a known voltage $\dot{E}_g$ (generator voltage) being maintained between the points A and B. Find expressions, in the abbreviated symbolic notation, for the current through

the impedance $r+jx$, and for the voltage across its terminals.* Solution : Determine an impedance Z_3 as the sum of the impedances $r+jx$ and r_2+jx_2. Let its reciprocal, or the equivalent admittance be Y_3, and the admittance of the circuit r_0, x_0, be Y_0. Then, the total admittance between the points C and D is $Y_3 + Y_0$. Find the corresponding impedance and add to it the impedance of the part r_1, x_1, of the circuit. This will give the total impedance, Z_{eq}, between A and B. To find the generator current, multiply the given voltage by the inverse operator, Y_{eq}, corresponding to this impedance. In order to find the load current and voltage, the current lost in the circuit r_0, x_0 must be sub-

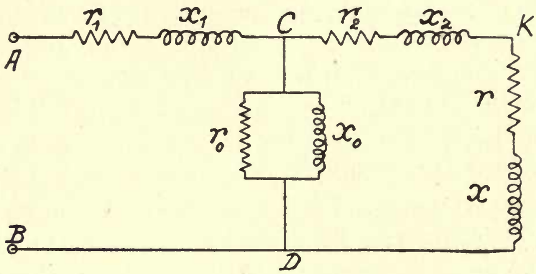

FIG. 9. A series-parallel combination of impedances.

tracted from the generator current, and the voltage drop in the part AK calculated. This is done exactly as in problem 3, except that impedances are used instead of resistances in calculating the drop, and the admittance Y_0 is used instead of the conductance g_0 in calculating the current in the branch r_0, x_0. In the abbreviated form the formulae are exactly analogous to those in problem 3, except for the notation. However, the actual substitution of complex quantities leads to long and complicated formulae, and is of no advantage for general purposes. It is preferable to perform these substitutions on a numerical example; see next problem.

*This diagram of connections represents what is known as the equivalent transformer diagram, the ratio of transformation being one to one. The impedance $r+jx$ represents the load, the other three impedances are intended to represent the various losses of voltage and current in the transformer itself. The diagram and the method of solution are analogous to those in problem 3, Chapter I.

PROB. 97. Apply the method outlined in the preceding problem to the following specific case: The voltage across AB is $\dot{E}_g = 2740 + j760$ volt; $r + jx = 25 + j10$ ohm; $r_1 + jx_1 = .7 + j1.2$ ohm; $r_2 + jx_2 = .8 + j1.5$ ohm; $r_0 = 760$ ohm; $x_0 = 540$ ohm. Ans. $\dot{I} = 94.86 - j16.94$ amp.; $\dot{E} = 2544 + j526$ volt.

PROB. 98. Deduce expression (55) for the inverse operator, without the use of the direct operator. Solution: Let it be required to find the projections of the current vector, knowing the projections of the voltage vector and the impedance of the circuit. The circuit is replaced by an equivalent parallel combination of a conductance g and susceptance b, as is explained in Sec. 15. The required current consists of two components: A working component, in phase with the voltage, equal to $g\,(e + je')$; and a wattless component, lagging behind the voltage by 90 degrees, and equal to $b\,(e + je')/j$; the division by j representing a rotation in the negative direction by 90 degrees. Adding the two component vectors of current and factoring out $e + je'$ we get $\dot{I} = (g + b/j)\,(e + je')$, or multiplying the numerator and the denominator of the term b/j by j,

$$\dot{I} = (g - jb)\,(e + je').$$

Therefore, $g - jb$ is the operator which converts the vector of the voltage into that of the current.

19. Power and Phase Displacement expressed through Projections of Vectors. Let an alternator supply a current $\dot{I} = i + ji'$ at a voltage $\dot{E} = e + je'$, and let it be required to determine the power output of the generator. The expression for the average power is $P = EI\cos\phi$, where ϕ is the phase displacement between I and E (see Sec. 10). The angle ϕ is the difference between the angles θ_1 and θ_2 which the vectors E and I respectively form with the reference axis. Hence, we have

$$\begin{aligned} P = EI\cos\phi &= EI\cos(\theta_1 - \theta_2) \\ &= E\cos\theta_1 \cdot I\cos\theta_2 + E\sin\theta_1 \cdot I\sin\theta_2. \end{aligned}$$

Remembering that $E\cos\theta_1$, $E\sin\theta_1$, etc., represent the projections of the given vectors on the axes of co-ordinates we have simply

$$P = ei + e'i'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

Another way of deducing expression (57) is to resolve the given vectors of current and voltage into their components along the axes of co-ordinates and to consider the contribution of each projection to the total power. The projections e and i, being in phase, give the power ei. Similarly, the vertical projections, e' and i', give the power $e'i'$. The vertical projection i' of the current gives an average power zero with the horizontal projection e of the voltage, the two being in phase quadrature. For the same reason the average power resulting from e' and i is equal to zero. Thus, $ei + e'i'$ represents total average power.

To find the phase displacement, or the power factor of the output, we write

$$\tan\phi = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2},$$

or

$$\tan\phi = \frac{e'/e - i'/i}{1 + e'/e \cdot i'/i}. \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Knowing $\tan\phi$, the angle itself or its cosine are found from the trigonometric tables; or, else, the power factor is caculated from the formula $\cos\phi = (1 + \tan^2\phi)^{-\frac{1}{2}}$.

Power factor can be also determined directly from the expression

$$\cos\phi = P/EI = (ei + e'i')/\left[(e^2 + e'^2)(i^2 + i'^2)\right]^{\frac{1}{2}},$$

but the calculation is more involved than when formula (58) is used.

The power calculated according to formula (57) sometimes comes out negative, if some of the projections of $\dot{E}$ and $\dot{I}$ are negative. The interpretation is that in this case the phase displacement between the current and the voltage is over 90 degrees, so that power is being supplied to the machine, instead of being delivered by it. Tan ϕ in formula (58) can also be negative, which means either that the current is leading, or that it is lagging by an angle larger than 90 degrees. The question is decided by reference to the sign of the power.

PROB. 99. The terminal voltage of an alternator is $5370 + j735$ volts, the line current is $173 - j47$ amps. Calculate the output

of the machine and the power factor of the load. Ans. 894.5 kilowatt.; 92 per cent, lagging.

PROB. 100. How should the vertical projection of the current in the preceding problem be modified in order that the power become zero? Ans. — 1264 amp.

PROB. 101. Let the line current in problem 99 be $-58 + j\,12$ amp. Explain the negative sign of power and the plus sign of $\tan\phi$; draw the vectors of the current and of the voltage.

PROB. 101a. A synchronous machine generates a voltage equal to $2300 - j\,50$ volts, and supplies a current, through an impedance $5 + j\,50$ ohms, to another synchronous machine generating a counter-e.m.f. of $2300 + j\,50$ volts. What is the power output of the first machine? Is the current leading or lagging? Ans. -4.545 kw. $\phi = 173° \, 3'$ lagging.

PROB. 102. A current of $350 - j\,75$ amp. is maintained through an impedance, the power output being 952 kw. at a power factor of 86 per cent lagging. Find the voltage across the impedance. Ans. $2930 + j\,987$ volts. Hint: Solve eqs. (57) and (58) together for the unknown projections e and e'.

PROB. 103. Solve problem 102 by calculating the value of the impedance, and multiplying the impedance by the current. Hint: power $= I^2 r$; $x = r \tan\phi$.

PROB. 104. The product $P_r = EI \sin\phi$ is called the *reactive power* (wattless power). Find its expression through the projections of the vectors of the voltage and the current, similar to eq. (57). Ans. $P_r = e'i - ei'$.

PROB. 105. Show that the relation $P^2 + P_r^2 = P_a^2$ exists between the true, the reactive and the apparent power; $P_a = EI$ is called the *apparent power*.

PROB. 106. Show that expression (58) can be also written in the form: $\tan\phi = P_r/P$.

PROB. 107. Solve problem 102 using the expressions for the reactive power derived in problems 104 and 106.

CHAPTER VI.

THE ELECTROSTATIC CIRCUIT.

20. **Physical Concept of the Electrostatic Field.** The fundamental phenomena of electrostatics are supposed to be known from physics. The purpose of this chapter is to give a somewhat different interpretation of these phenomena, and to deduce numerical relations which are of importance in electrical engineering. The reader is advised to consult also Chapter XXI, on Electrostatic Capacity, in the author's Experimental Electrical Engineering (Chapter VI in the first edition).

Let a source E of *continuous* electromotive force (Fig. 10) be connected to two parallel metallic plates A and B, the combination of which is commonly known as a *condenser*. The plates are separated from each other by air, or by some other non-conducting material. When the key K is closed, a certain quantity of electricity, Q, flows from the battery to the plate A, and from the plate B back to the battery. This quantity can be measured on the ballistic galvanometer shown in the circuit. Within a very short time the difference of potential between the plates becomes equal to that of the battery, so that the flow of current stops. The plates are then said to be charged with certain quantities of electricity, $+Q$ and $-Q$.

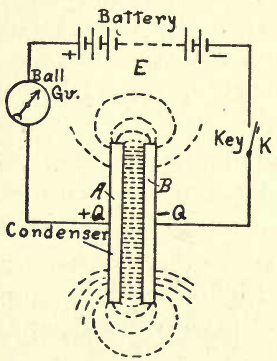

FIG. 10. A plate condenser completing a direct-current circuit.

From a modern point of view, it is preferable to consider electricity as an incompressible fluid. Therefore, no charges are accumulated on the plates, but a quantity of electricity Q is displaced in the circuit, including the layer of the insulation or *dielectric* between the plates. This displacement is accompanied in the dielectric by a stress, similar in some respects to a mechanical stress in an elastic body. The direction of the electric stress and of the lines of displacement of electricity are

shown in the figure by dotted lines. When the key is opened, the condenser remains charged, since the stress and the displacement can be equalized only in a closed circuit. To discharge the condenser its plates must be connected by a conductor. Then the deflection of the ballistic galvanometer is equal and opposite to that during the charge, and the electric energy stored in the condenser is dissipated in the form of heat by the current.

The difference between a dielectric and a conductor is that the resistance of the former to the passage of electricity is of an elastic nature; that is to say, the stress can be relieved and the stored energy can be returned to the circuit. On the contrary, the resistance to the flow of electricity in a conductor has the nature of friction. The energy spent is converted into Joulean heat and cannot be restored.

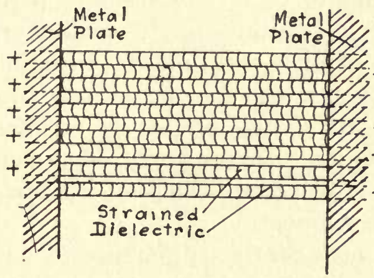

FIG. 11. Electric stress and displacement in a dielectric, represented by filaments of positive and negative electricity, shearing past one another.

When the voltage E is sufficiently high, the displacement of electricity in the dielectric reaches a limit at which the material is "broken down", and a disruptive discharge, or a spark, occurs between the edges of the plates. With solid and liquid insulating materials, such as glass, oil, mica, etc., larger charges or displacements Q are obtained in the same condenser, with the same voltage E.

These and other phenomena of electrostatics are accounted for by assuming a certain hypothetical structure of dielectrics. Namely, a dielectric is supposed to contain in its natural, unelectrified state both positive and negative electricity, combined so as neutralize each other. The process of electrification (Fig. 11) consists in displacing positive electricity towards one

electrode or plate, and negative electricity towards the other plate. This separation, caused by the applied electromotive force, produces a stress in the dielectric. While the actual mechanism of the phenomenon is unknown, Fig. 11 give a simple and possible picture of it.

The positive and the negative electricity are represented by alternate threads or filaments. The electromotive force, according to this analogy, pulls positive threads in one direction, and negative ones in the opposite direction. The connection between adjacent threads must be assumed to be of an elastic nature, so that this pull produces a shearing stress along the periphery of the threads; this stress opposes their displacement. The displacement reaches its limit when the shearing stress just balances the applied electromotive force. The same filaments, or something corresponding to them (for instance paths of electrons), must be assumed to continue in the conductors, but there the resistance to the motion is of the nature of friction. This friction, as in ordinary fluids, must be supposed to be proportional to the velocity of motion. Therefore, it is manifested only when electricity is in motion, in other words, when a current is flowing. This assumption is consistent with Ohm's law for conductors, and hence may be adopted.

A stress, and consequently a difference of potential, exists in a dielectric when the threads have been displaced from their natural relative position, though the threads themselves may be stationary in the new position. The stress is relieved by removing the electromotive force and by providing a closed path of low resistance, through which the threads can again be relatively displaced into their natural position. What according to the old theory is considered as charges on condenser plates, is but a displacement in the intervening dielectric. The plates are only the boundaries of this dielectric, on which boundaries the phenomenon apparently takes place. When a current is flowing through the metallic part of the circuit an equal displacement current is established in the dielectric, which thus completes the circuit.

When an *alternating* voltage is applied at the terminals of a condenser, the displacement in the dielectric changes continually in its magnitude and direction; consequently, it gives rise to an alternating current in the metallic part of the circuit. This is

called the charging or the capacity current. This current leads the alternating voltage in phase by 90 degrees, as may be seen from the following considerations: When the voltage has reached its maximum, the charging current is zero, because at the crest of the wave the voltage and the displacement remain practically constant for a short period of time. As soon as the voltage begins to decrease, the current begins to flow in the direction opposite to that of the applied voltage, because the elastic reaction of the dielectric is now larger than the applied electromotive force. At any instant the current, or the rate of flow of electricity, is proportional to the rate of change of the applied voltage. But if the applied voltage varies according to the sine law, the rate of variation is also represented by a sine function, differing in phase by 90 degrees from the original function $[d(\sin x)/dx = \cos x = \sin(90 + x)]$. It is well worth the student's while to think over this phase relation between the charging current and the voltage more in detail. That there must be a displacement of 90 degrees between the voltage and the current follows also directly from the assumed elastic structure of the dielectric. Namely, the energy is supposed to be periodically stored in the dielectric and given up again without any loss. Hence, the average power must be zero, and the current must be wattless. See also Art. 24 below.

A hydraulic analogy shown in Fig. 12 may assist the student in the understanding of the mechanism of the electrostatic circuit. *A* is a pump which corresponds to the source of electromotive force in Fig. 10. The pipes *B* and *C* represent the leads to the condenser, or the metallic parts of the circuit. The cylinder *D* corresponds to the condenser, and the elastic partition *K* is analogous to the dielectric. Let the pipes and the cylinders be filled with water, and let the piston in *A* be in its middle position, the partition *K* being not stressed. Let the stop-cock *M* be open, and the stop-cock *N* be closed. When a pull to the left is exerted upon the piston rod and it is made to move, the water in the system is displaced, and

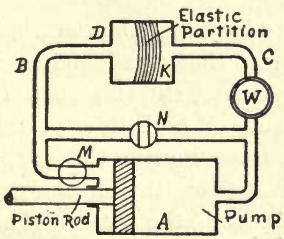

FIG. 12. A hydraulic analogue of an electrostatic circuit.

the elastic partition K is strained, as is shown in the figure.* With a given pull, or a given electromotive force, the movement stops when the pull is balanced by the elastic reaction of the partition. The charge, or the total displacement, is represented by the amount of the water shifted ; it can be measured on the water meter W, which thus takes the place of the ballistic galvanometer.

If the pipes are frictionless, the analogy can be followed still further ; namely, the phase difference in time between the pull and the velocity of the water is 90 degrees, the latter leading the pull. Assuming the motion of the piston to be harmonic, the velocity of the flow of water is at its maximum, when the piston is at the center of its stroke. The required pull is equal to zero at this moment, because the elastic partition is in its middle, unstrained position. At the end of the stroke, the velocity is zero, but the pull is at its maximum, because the partition is strained to its extreme position, and exerts its maximum elastic reaction.

Substituting another partition, made of a more yielding material (material possessing higher *permittivity*) a larger displacement is produced with the same pull; this corresponds to the case when some solid or liquid dielectric is substituted for the air. The quantity of electricity displaced per unit pull (that is per unit voltage) is called the capacity of the condenser. Thus we see that the capacity of a condenser is increased by the substitution of a material possessing higher permittivity.

Closing the stop-cock M corresponds to breaking the electrical circuit of the condenser. It will be seen that the condenser remains charged. To discharge the condenser, the stop-cock N must be opened ; this equalizes the pressure on both sides of the elastic partition. Since water possesses some inertia the partition does not stop in its middle position during the discharge, but the momentum of the water carries it beyond the center. The electromagnetic inertia of the electric current produces a similar effect, and we thus have an explanation of the oscilatory character of the electric discharge. During this discharge the energy

*The diameter of the cylinder K must be conceived to be many times larger than that of the pump A.

is alternately transformed into the potential energy of the dielectric stress, and into the kinetic energy of the magnetic field. The oscillations of the partition are gradually damped out by the frictional resistance of the pipes. In the electric circuit oscillations are damped by the ohmic resistance in the conducting parts of the circuit.

The student can follow this analogy still further and explain the natural period of vibration, current and voltage resonance, and the effect of a resistance in series and in parallel with a condenser, etc.

21. Dielectric Flux Density and Electrostatic Stress (Voltage Gradient). Referring again to the uniform electrostatic field in Fig. 10, consider a cube in the dielectric, one square centimeter in cross-section, and one centimeter long in the direction of the lines of force. Let a quantity of electricity Q be supplied by the battery, as shown by the ballistic galvanometer; then the same quantity of electricity must be displaced in the dielectric. Neglecting a small displacement at the edges and at the outside surfaces of the plates, it may be said that the whole quantity Q is uniformly displaced between the plates. Therefore, if the area of each plate is equal to A square centimeters, the displacement through the cube under consideration is equal to

$$D = Q/A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

Since Q is the total electrostatic flux, D is naturally called the *dielectric flux density*. If Q is measured in coulombs, D is expressed in coulombs per square centimeter. In practice Q is measured in microcoulombs, and D is expressed in microcoulombs per square centimeter, or per square inch. It will be remembered that a coulomb is the quantity of electricity which a current of one ampere supplies during one second. One microcoulomb is one millionth part of a coulomb.

By analogy with the shearing stress in the theory of elasticity, the stress in the dielectric (Fig. 11) can be defined as the sum of the shearing forces between the positive and the negative filaments within a unit cube. When the condenser is fully charged, the sum of these shearing forces is balanced by the electromotive force between the plates. Thus, if the distance between the plates is l centimeters and the voltage is E, we have $E = l \cdot F$,

where F is the electrostatic stress, or the force per unit length of the unit filament (the force per unit cube). In other words,

$$F = E/l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

or, *the electrostatic stress in a uniform field is equal to the voltage gradient per unit length of the field.* F is expressed in volts (or in kilovolts) per centimeter, or per inch.*

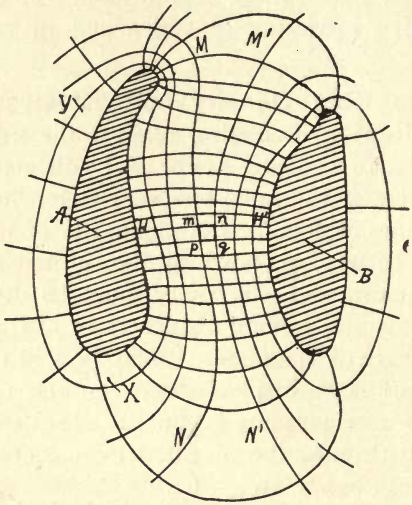

FIG. 13. A non-uniform electrostatic field, represented by lines of displacement and by equipotential surfaces.

Equations (59) and (60) connect two quantities, the voltage E and the charge Q, with the flux density D and the stress F. The first two quantities characterize the circuit as a whole, the two latter characterize the dielectric proper. It will be seen that these equations are analogous to the eqs. (11) and (12) in Chapter I. The expression "electric intensity" used there is the same as the expression "stress" used here. It is also proper to call F the voltage gradient.

*The voltage E is the cause of a state of stress in a dielectric of thickness l. Since, by supposition this state is uniform throughout, it takes a voltage E/l to produce this stress per unit thickness of dielectric. This quantity is denoted as the stress F. With this interpretation it is not necessary to define or describe by analogies the nature of the electric stress.

When the field is non-uniform, as in Fig. 13, an infinitesimal parallelopiped *mnpq* must be considered, instead of a unit cube. Let the cross-section *mp* of this parallelopiped, perpendicular to the direction of the field, be dS, and the length *mn* in the direction of the field equal dl. Let the displacement through the parallelopiped be dQ. Then the dielectric flux density, or the displacement per unit area, is

$$D = dQ/dS. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

Again, let the voltage between the opposite sides *mp* and *nq* of the parallelopiped, in the direction of the field, be dE. Then, the stress, or the voltage gradient, at the center of the parallelopiped is equal to

$$F = dE/dl. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

Equations (61) and (62) must be used in place of eqs. (59) and (60) when the field is non-uniform. They give D and F as functions of Q and E. If it is desired to express Q and E in terms of D and F, the foregoing equations must be integrated. Integrating eq. (61) we get

$$Q = \int_0^S D \cdot dS, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

where the integration has to be extended over a complete closed equipotential surface, such as XY, surrounding one of the condenser plates. This follows from the fact that dS is selected in each elementary parallelopiped in a plane perpendicular to the displacement, and hence lies in an equipotential surface. Integrating eq. (62) we obtain

$$E = \int_0^l F \cdot dl, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

where the integration is to be performed along a line of force, such as HH', between the positive and negative plates of the condenser.

The last two relations are expressed in words by saying that the total displacement Q is a surface integral of the dielectric flux density, while the voltage E is a line integral of the dielectric stress. A little consideration will show that these relations are almost self evident; at least that they follow directly from

the definitions of the quantities concerned, and from the assumed structure and properties of dielectrics. Compare also with Article 5 in Chapter I.

PROB. 108. A condenser (Fig. 10) consists of two metal plates, 50 × 70 cm. each, in contact with a glass plate 3 mm. thick between them. When a continuous voltage of 2400 v. is applied to the condenser, the ballistic galvanometer shows a charge of 17.1 microcoulombs. What is the dielectric flux density and the stress in the glass? Ans. $D = 17.1/(50 \times 70) = 4.885 \times 10^{-3}$ mc./cm.^{2}; $F = 2.4/.3 = 8$ kv./cm.

PROB. 109. What is the flux density and the voltage gradient in the preceding problem, expressed in the English system? Ans. $D = 31.52 \times 10^{-3}$ mc./inch^2; $F = 20.32$ kv./inch.

PROB. 110. A certain kind of oil-cloth insulation is broken down between two plates at a voltage gradient of about 20 kv. per mm. (maximum instantaneous value). What should be the thickness of this insulation between two plates to stand 45 effective kilovolts, at a factor of safety of 5? Ans. About 5/8 inch.

PROB. 111. A single-core cable (Fig. 14) receives a charge of .0019 coulombs per mile when a continuous voltage of 12 kv. is applied between the core and the sheathing. The core consists of No. 4 solid conductor B. & S., the insulation is 3/8 inch thick. Determine the law according to which the dielectric flux density varies within the insulation, and the extreme values of the density. Ans. Equilateral hyperbola, $D = 4.77 \times 10^{-3}/x$ mc. per square inch, where x is the radius of the layer under consideration, in inches; $D_{max} = .0467$; $D_{min} = .01$.

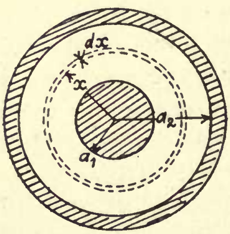

FIG. 14. Cross-section of a single-core, lead-covered cable.

PROB. 112. The cable insulation in the preceding problem breaks down at a maximum instantaneous flux density of .4 mc./inch^2. At what alternating voltage between the core and the sheathing does this occur, and which layer of insulation breaks down first? Ans. At about 73 effective kilovolts, near the core. Hint: Flux density is proportional to the voltage.

22. Permittivity and Dielectric Strength. As long as an insulating material is stressed well below the point at which a disruptive discharge occurs, the dielectric flux density is proportional to the stress, that is to say

$$D = \kappa F. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Here κ is a coefficient of proportionality called the *permittivity* of the dielectric. This law expresses the fact that the effect is proportional to the cause. It is analogous to Hooke's law for elastic materials: displacement corresponds to strain, and $1/\kappa$ corresponds to the modulus of elasticity. Eq. (65) is also analogous to Ohm's law, in the form given by eq. (10). To conductivity γ corresponds here the permittivity κ. Conductivity indicates the ease with which electricity can *flow* through a metal or imperfect insulator; permittivity shows the ease with which electricity can be *displaced* through a dielectric. One indicates frictional resistance to motion, the other an elastic resilience.

For air the permittivity in the metric system is equal to

$$\kappa_a = .08842 \times 10^{-6} \quad . \quad . \quad . \quad . \quad . \quad (66)$$

microcoulombs per volt, per square centimeter cross-section, per centimeter length. In other words, in an air condenser, with the distance between the plates equal to one centimeter, the displacement per one square centimeter of cross-section of the dielectric is equal to $.08842 \times 10^{-6}$ microcoulombs, when the difference of potential between the plates is one volt. The dimensions of the plates are assumed to be very large as compared to the distance between them, so that the influence of the edges can be neglected. See Note at the end of the chapter.

The value of permittivity of air in the English system is

$$\kappa_a = .2244 \times 10^{-6} \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

in microcoulombs per one square inch of the area of the dielectric, per volt, per inch of the length of the lines of force.

For other substances, it is convenient to use *relative* values of permittivities, referred to that of air as unity.* In this way, the relative properties of various dielectrics are made more evident, and the necessity is obviated for tabulating very small

*The older name for relative permittivity is *specific inductive capacity.*

quantities, like those in eqs. (66) and (67). The values of relative permittivities for some materials are given in the table below, in the second column.

Substance.	Relative permittivity, or specific inductive capacity K.	Rupturing voltage gradient F_{max}, in eff. kv. per mm.	Rupturing values of dielectric flux densities, D_{max}, in eff mc. per sq cm.
Air	1	2.7	.0024
Glass, different kinds	3—8	6—8	.016—.056
Mica, natural and built up	5—8	17—28	.075—.200
Porcelain, D.C.	5.3	——	——
Porcelain, A.C.	4.4	9—16	.035—.062
Rubber, pure	2.2	20—25	.039—.049
Rubber, vulcanized	2.7	15—20	.036—.048
Transformer oil	2—2.2	10	.018—.019
Vacuum	.99	——	——

NOTE: The data given in the table indicate the order of magnitude rather than accurate values. The last two columns contain the *effective* values of voltage gradient and of displacement, under the supposition of a sine-wave law of variation of the applied voltage.

It will be seen from the table, that the permittivities of solid and liquid dielectrics are larger than those of air; in other words, they are more yielding to the electric stress than the air. This does not mean, however, that they break down at a lower voltage gradient than the air. On the contrary, the third and the fourth columns show that the dielectrics commonly used in electrical engineering are considerably stronger electrically than the air, in that they can stand several times the displacement at which the air breaks down.

The above-mentioned fact indicates that the permittivity alone is not sufficient to characterize an insulating material for practical uses, but that its "dielectric strength" must also be known. There does not seem to be any relation between these two coefficients: One indicates the elasticity of the material, the other its ultimate strength. In this respect, they are analogous to the modulus of elasticity and to the rupturing load in the mechanics of materials. Air, from an electrical point of view, may be said to be a material of great stiffness, but one which breaks at a comparatively small elongation. On the contrary, mica is comparatively yielding, but can stand a very large elongation before it is ruptured; so that, in spite of larger permittivity, a much

higher stress is required to rupture mica than air. The student is advised to make clear to himself these two separate properties of dielectrics: A rational design of high-tension insulation 'depends essentially upon a distinct understanding of these properties.

Dielectric strength is properly given as the critical flux density, D_{max}, but for practical purposes it is more convenient to express it as the voltage gradient, F_{max}, at which the dielectric is broken down. When a dielectric is used in the form of thin sheets having a comparatively large radius of curvature, the flux density, and, consequently, the voltage gradient are uniform throughout, so that $F_{max} = F_{ave}$. When, however, the dielectric is thick as compared to its radius of curvature, as for instance in high-tension machines, or when air or oil are tested between two spherical terminals, the use of the average voltage gradient $F_{ave} = E/l$ leads to wrong results. The only proper way in this case is to calculate the dielectric flux density where it is at a maximum, and to see that this density, D_{max}, and the corresponding value of the voltage gradient, $F_{max} = D_{max}/\kappa$, do not exceed the critical values, determined from previous tests.

PROB. 113. Show how the value of κ_a in eq. (67) is derived from that in eq. (66). Ans. $.08842 \times 10^{-6} \times (2.54)^2/2.54 = .2244 \times 10^{-6}$.

PROB. 114. Show how the values in the last column of the table are derived from those in the two preceding columns. Ans. $D_{max} = .08842\, KF_{max} \times 10^{-2}$.

PROB. 115. What is the relative permittivity (specific inductive capacity) of the glass in problem 108? Ans. 6.9.

PROB. 116. A certain material stood about 82 kv. in a layer 3.7 mm. thick. What voltage gradient can be allowed in this material at a factor of safety equal to 2? Ans. 11 kv. per mm. thickness.

23. Permittance, or Electrostatic Capacity. The total displacement produced in the dielectric of a given condenser is proportional to the voltage at the terminals of the condenser, because voltage and displacement appear as cause and effect. Hence, we have the relation

$$Q = CE, \qquad (68)$$

where the coefficient of proportionality C is properly called the *permittance* of the condenser. The larger C the larger the total displacement Q for a given voltage, or the more yielding is the condenser. The older name for permittance is electrostatic capacity. The name permittance is much more preferable, for the following reasons: (1) It suggests the true action of the dielectric, while the word capacity was intended to characterize the metallic plates of the condenser according to the old view, that is to say, with regard to their ability to store or to "condense" electricity.* (2) Permittivity and permittance stand in the same relation to each other as resistivity and resistance, or as permeability and permeance, conductivity and conductance, etc., so that a uniformity of notation is preserved.

The unit of permittance or capacity in the volt-ampere-ohm system is the *farad*. A condenser is said to have its permittance equal to one farad when one coulomb of electricity is displaced in its dielectric per volt of pressure between its terminals. The farad is by far too large a unit for practical purposes; therefore, permittance is usually measured in microfarads. The permittivity κ which has been defined above in terms of microcoulombs per volt, per cubic unit of dielectric, now can be more simply defined as microfarads per cubic centimeter, or per cubic inch.

When the lines of force in a prismatic slab of dielectric are parallel to each other and parallel to one of the principal dimensions of the slab, its permittance is expressed by a formula similar to that for the conductance of a prismatic conductor. Namely,

$$C = \kappa A/l. \qquad (69)$$

This follows directly from the assumed structure of the dielectric. The reasoning is the same as in the deduction of the corresponding expression for the conductance: When $A = 1$, and $l = 1$, the permittance is by definition equal to permittivity κ. If the cross-section is increased A times, as many times more electricity is displaced with the same voltage between the plates. If the length is increased l times, the voltage gradient E/l be-

*An incompressible substance like electricity evidently cannot be condensed. To complete the rational nomenclature, the word "condenser" should be replaced by "permittor" or "elastor", to correspond with "conductor" and "resistor."

comes l times smaller, and consequently l times less electricity is displaced per unit area.

Applying the foregoing formula to the condenser in problem 108, its permittance is found to be equal to

$$C = .08842 \times 10^{-6} \times 6.9 \times 50 \times 70/.3 = .00712 \text{ mf.},$$

where the relative permittivity of the glass, $K = 6.9$, is supposed to be given.

When the dielectric in a condenser is not prismatic in form, which is mostly the case, its permittance is determined by dividing the dielectric into infinitesimal layers, or into a large number of very thin layers and threads (Fig. 13). The permittance of each element can be expressed by the formula (69). The permittance of the whole dielectric is obtained by combining the layers and the threads in series and in parallel, as the case may be. The procedure is analogous to that of finding the resistance of a conductor of variable cross-section (see Sec. 5).

This leads to the question of combining permittances, or capacities, in series and in parallel. When two or more permittances are connected in parallel between the same electrodes, the combined, or the equivalent permittance is equal to the sum of the component permittances. The proof is exactly the same as in the case of conductances in parallel, Sec. 2; thus, we have

$$C_{eq} = \Sigma C. \quad \quad (70)$$

When permittances are connected in series, the equivalent permittance is smaller than any one of the component permittances, because the voltage drop is distributed over a greater length, and the stresses become smaller. As in the case of conductances, we have

$$C_{eq}^{-1} = \Sigma C^{-1}. \quad \quad (71)$$

For a formal proof of eqs. (70) and (71) see Art. 430 (Art. 121 in the first edition) of the author's Experimental Electrical Engineering.*

*When permittances are connected in series, it is more convenient to treat the reciprocal C^{-1} of each permittance as a new physical quantity, which can be properly called the *elastance* of the dielectric. Then eq. (71) simply expresses that the equivalent elastance is equal to the sum of the elastances in series; this is analogous to the addition of resistances in series. Conductance and permittance show the ease with which electricity can flow

PROB. 117. A condenser which has a permittance of 10 mf. is connected to a direct-current magneto, the speed of which is increased at a uniform rate, so that the voltage rises at a rate of 1.7 volts per sec. Calculate the charging current. Ans. 17 microamperes. NOTE. This is the principle of an apparatus used for measuring the acceleration of railway trains.

PROB. 118. A .5 mf. mica condenser is to be made out of sheets of mica 12 × 25 cm., .3 mm. thick, and coated on one side with a very thin film of silver. How many sheets are required? The relative permittivity of the mica is equal to about 6. Ans. About 96 sheets, 48 sheets in parallel per terminal.

PROB. 119. Let the dielectric in problem 108 consist, instead of glass, of three layers of different materials. Let the thicknesses of these layers be 1.2, .7, and 1.1 mm., and let the corresponding values of relative permittivities be 2, 3, and 5. What is the capacity of the condenser? Solution: The layers being divided by equipotential surfaces, infinitely thin, insulated metal plates may be imagined between the separate materials, without changing in any way the stresses or the displacements. The problem is thus reduced to that of calculating the equivalent permittance of three permittances in series. The component permittances, per unit area of the plates, are: $2\kappa_a/.12$; $3\kappa_a/.07$; $5\kappa_a/.11$. Hence, according to eqs. (70) and (71), we have that $C = \kappa_a \times 50 \times 70/(.12/2 + .07/3 + .11/5) = .00294$ mf.

PROB. 120. The condenser described in the preceding problem is subjected to a difference of potential equal to 10 kv. What are the voltage gradients (stresses) in the separate dielectrics?

---

or be displaced; their reciprocals, resistance and elastance, show the degree of difficulty to the passage of electricity. Similarly, the coefficient $1/\kappa$ should be called the *elastivity* of the medium, being analogous to resistivity ρ, and to the modulus of elasticity in mechanics. Elastance of dielectrics could be properly measured in units called *darafs*, one daraf being the reciprocal of one farad; it is therefore designated by the same name spelled backwards (same as ohm and mho). As the farad is too large a unit, so the daraf is too small a unit for practical purposes; elastances are conveniently measured in megadarafs, same as insulation resistance is measured in megohms. One megadaraf is the reciprocal of one microfarad. There is little doubt that with the extended use of extra-high voltages there will be more and more need for electrostatic calculations, and for a scientific design of insulation; then some such names and units will have to be agreed upon.

Solution : Voltage drops across condensers in series are to each other as their elastances (note analogy to the case of resistances in series). Hence, denoting the drops by x_1, x_2, x_3, we have $x_1 : x_2 : x_3 = .12/2 : .07/3 : .11/5$. Besides, $x_1 + x_2 + x_3 = 10$. Solving these equations we find $x_1 = 5.7$ kv.; $x_2 = 2.21$ kv.; $x_3 = 2.09$ kv. The corresponding potential gradients are 4.75, 3.16, and 1.9 kv. per mm. of thickness.

PROB. 121. Deduce a general expression for the capacity of a single-core, lead-covered cable (Fig. 14), L miles long. The capacity is understood here as the permittance between the conductor and the sheathing. Solution : The elastance (reciprocal of permittance) of an infinitesimal concentric layer of thickness dx and of the length l inches is $dx/(\kappa \cdot 2\pi x l)$, so that the elastance of the whole dielectric is

$$\int_{a_1}^{a_2} dx/(\kappa \cdot 2\pi x l) = (1/2\pi\kappa l) \text{ Ln } a_2/a_1. \quad . \quad . \quad (72)$$

The permittance of the cable is the reciprocal of its elastance. Using common logarithms, we get

$$C = 2\pi\kappa l/[2.3026 \log a_2/a_1]. \quad . \quad . \quad . \quad . \quad (73)$$

In the English system $\kappa = .2244\, K \times 10^{-6}$ where K is the relative permittivity of the insulation used. The length in inches $l = 12 \times 5280 L$. Substituting, we get

$$C = .0388 KL/\log (a_2/a_1), \text{ (in mf.)}. \quad . \quad . \quad . \quad (74)$$

PROB. 122. What is the capacity per mile of the cable given in problem 111, and what is the specific inductive capacity of the dielectric? Ans. $C = .1583$ mf./mile ; $K = 2.73$.

PROB. 123. What is the maximum and the average stress in the cable in problem 111 (use answers to problems 111 and 122)? Solution : $F_{max} = D_{max}/K = .0467/(.2244 \times 10^{-6} \times 2.73) = 76.2$ kv./inch. $F_{ave} = 12/.375 = 32$ kv./inch. NOTE : The results derived in this problem are of great practical importance. They corroborate the statement made above that unless the thickness of a dielectric is small as compared to the radius of curvature, the average stress gives no idea of the maximum flux density at the most dangerous point.

PROB. 124. What is the general expression for the stress in the insulation, in the cable shown in Fig. 14, at a distance x from the center, when the voltage between the core and the sheathing is equal to E? Solution: The dielectric flux density, and consequently the stress, is inversely proportional to the distance from the center of the cable. The stresses are also proportional to the applied voltage. Hence, $F = NE/x$, where N is a constant. This constant is determined from the condition that the applied voltage is the line integral of the stress, see eq. (64). We have

$$E = \int_{a_1}^{a_2} F dx = NE \int_{a_1}^{a_2} dx/x = NE \operatorname{Ln} (a_2/a_1).$$

Or, $N = (\operatorname{Ln} a_2/a_1)^{-1}$, so that $F = E/(x \operatorname{Ln} a_2/a_1)$.

PROB. 125. Check the solution of problem 123 using the result of the preceding problem.

PROB. 126. Show by actual calculation that in the foregoing cable, the *maximum* stress in the dielectric is reduced by using conductor No. 1 B. & S. instead of No. 4, with the same diameter of the sheathing. This result is obtained in spite of the fact that the insulation becomes thinner, and consequently has a larger *average* stress.

PROB. 127. Referring to the preceding problem, show that it is of advantage to make the ratio of a_2/a_1 about equal to ϵ, where $\epsilon = 2.71828 \ldots$ is the base of natural logarithms. When the diameter of the conductor is further increased, so that the ratio of a_2/a_1 becomes less than ϵ, a further increase in the diameter of the core increases the maximum stress, instead of reducing it. Solution: The stress at the core is

$$F_1 = E/\left[a_1 \operatorname{Ln} (a_2/a_1) \right],$$

according to problem 124. F_1 reaches its maximum with a variable a_1 when $dF_1/da_1 = 0$. Differentiating, we get

$$dF_1/da_1 = E\left[1 - \operatorname{Ln} (a_2/a_1)\right] / \left[a_1 \operatorname{Ln}(a_2/a_1)\right]^2 = 0;$$

whence, $1 - \operatorname{Ln} (a_2/a_1) = 0$, or $a_2/a_1 = \epsilon$.

PROB. 128. The insulation of a cable consists of two layers, the outside radius of the inner layer being b. The relative permittivities of the materials are K_1 and K_2. Extend the expression

(74) for the capacity to this case. Solution: The two permittances

$$C_1 = .0388\, K_1 L / \log\,(b/a_1)$$

and

$$C_2 = .0388\, K_2 L / \log\,(a_2/b)$$

are in series. Therefore, the resultant permittance, according to formula (71), is

$$C = .0388\, L \left\{ \Big[\log\,(b/a_1)\Big]/K_1 + \Big[\log\,(a_2/b)\Big]/K_2 \right\}^{-1}.$$

The result can evidently be extended to any number of concentric layers of insulation.

Note to page 67. The author considers the above-given value of κ for air to be an experimental coefficient, in the same sense in which other properties of materials are characterized by experimental coefficients. This is because for an engineer the volt and the ampere are arbitrary units established by an international agreement, no matter what their relation to the so-called absolute units.

The value of κ can be calculated theoretically, assuming the ratio between the electrostatic and the electromagnetic units to be known. Namely, in the absolute electrostatic system of units a plate condenser having an area of A sq. cm. and a distance between the plates equal to l cm., has a capacity equal to $A/4\pi l$. The factor 4π enters on account of an unfortunate selection of the expression for Coulomb's law; namely it should have been $ee'/4\pi r^2$, instead of ee'/r^2. In the absolute electromagnetic units the same capacity is equal to $(A/4\pi l)\,(3 \times 10^{10})^{-2}$, where 3×10^{10} is the velocity of light in centimeters per second. To obtain the result in microfarads, the foregoing expression must be multiplied by 10^{15}. On the other hand, the same capacity, expressed in the rational units, is $= \kappa A/l$. Equating the two expressions, gives $\kappa = 10^{-5}/(9 \times 4\pi) = .08842 \times 10^{-6}$ microfarads (microcoulombs per volt) per square centimeter cross-section, per centimeter length.

The fact that κ can be expressed through the velocity of light does not make κ the less an empirical coefficient, because the velocity of light itself is determined experimentally. As a matter of fact, one of the ways in which the velocity of light is determined consists in calculating it indirectly from the value of κ obtained from measurements.

CHAPTER VII.

THE ELECTROSTATIC CIRCUIT.

(*Continued*).

24. **Capacity in Alternating-Current Circuits.** The effect of electrostatic capacity in alternating-current circuits is discussed in detail in Arts. 436 to 445 of the author's Experimental Electrical Engineering (Arts. 127 to 136 in the first edition.) It is shown there that an electrostatic capacity or permittance gives rise to a charging current, which leads the voltage at the terminals of the condenser by 90 degrees in phase. The value of the charging current is

$$I = E \cdot 2\pi f \cdot C \times 10^{-6} \quad . \quad . \quad . \quad . \quad . \quad (75)$$

where f is the frequency in periods per second, and C is the capacity in microfarads. The product $2\pi fC \times 10^{-6}$ has the dimension of a mho, because it is equal to the ratio of a current to a voltage. By analogy with the inductive susceptance introduced in Art. 15, the product $2\pi fC \times 10^{-6}$ is called the *capacity susceptance*, and is also measured in mhos. Some authors call it capacitance.

The inductive susceptance causes the current to lag 90° behind the applied voltage, while the capacity susceptance makes it lead by 90°. Therefore, in the analytical treatment, given in Chapter V, it is necessary to consider the susceptance negative, when it is caused by a condenser, and positive when it is caused by a reactive coil. We have, therefore,

$$b = -2\pi fC \times 10^{-6}. \quad . \quad . \quad . \quad . \quad . \quad (76)$$

The reciprocal of a susceptance is a reactance, and we have, as in Art. 11 that the capacity reactance

$$x = -10^6/2\pi fC. \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

It must be remembered that a reactance is numerically equal to the reciprocal of a susceptance only when no ohmic resistance is present. Otherwise eqs. (45) and (46) must be used. The use of the capacity susceptance and of the capacity reactance is made clearer in the solution of the examples that follow.

PROB. 129. A condenser of 7.3 mf. permittance is connected across a 500 v., 60 cycle supply. What is the susceptance and the charging current? Ans. $b = -.002754$ mho; $I = j1.377$ amp. the voltage being the reference vector.

PROB. 130. The condenser in the preceding problem is shunted by a non-inductive resistance of 750 ohm. Find the total current and the power factor. Solution: The current through the resistance is $= 500/750 = .6667$ amp.; $\tan \phi = 1.377/.6667 = 2.065$; $\cos \phi = 43.58$ per cent (leading). Total current $= .6667/.4358 = 1.53$ amp.

PROB. 131. The condenser and the resistance in the preceding problem are connected in series, instead of in parallel. What is the equivalent parallel combination? Ans. $C_p = 1.387$ mf.; $r_p = 926$ ohm.

PROB. 132. A magnetic reactance of 65 ohm is connected in parallel with a capacity of 73.6 mf., across a 2200 v., 25 cycle circuit. Determine the total current, and the component currents, through the reactance and through the condenser. Ans. $33.85 - 25.44 = 8.41$ amp. (lagging). This is a case of partial current resonance, the total current being smaller than one of its components.

PROB. 133. The condenser and the reactance coil given in the preceding problem are connected across the same line in series, instead of in parallel. Find the total current and the component voltages. Ans. 102.3 amp. (leading); $2200 = 8850 - 6650$ volts. This is a case of partial voltage resonance, the voltage drop across one of the two devices being larger than the applied voltage.

PROB. 134. The voltage at the receiver end of a 25 cycle single-phase transmission line is $45 + j57$ kv.; the load current is $178 + j69$ amp. The impedance of the line is $32 + j68$ ohm; the capacity of the line is 4.24 mf. Calculate the generator current and voltage. For the purposes of calculation one half of this capacity can be assumed to be connected across the generator end of the line, the other half across the receiver end. Solution: The capacity susceptance at the receiver end of the line in mhos is $-2\pi \times 25 \times 2.12 \times 10^{-6} = -.333 \times 10^{-3}$. The corresponding charging current is $j.333 \times 10^{-3} (45000 + j57000) = -19 + j15$ amp.

Consequently, the total line current is $159 + j84$ amp. The line drop is $(159 + j84)(32 + j68) = -624 + j13500$ volt. Generator voltage $= 44.38 + j70.5$ kv. Charging current at the generator end $= j.333\,(44.38 + j70.5) = -23.5 + j14.79$ amp. Generator current $= 135.5 + j98.8$ amp.

PROB. 135. Indicate the general method of solution of the preceding problem in the abbreviated symbolic notation. Solution: Let the capacity susceptance at each end of the line be b, where b is a negative quantity. Let the receiver or the load voltage be $\dot{E}_L$. Then, the charging current at the receiver end is $\dot{I}'_C = -jb\dot{E}_L$, and the line current is $= \dot{I}_L + \dot{I}'_C$. If the line impedance is Z, the generator voltage $\dot{E}_G = \dot{E}_L + Z\,(\dot{I}_L + \dot{I}'_C)$. The charging current at the generator end is $\dot{I}''_C = -jb\dot{E}_G$. Generator current $\dot{I}_G = \dot{I}_L + \dot{I}'_C + \dot{I}''_C$.

25. Energy in the Electrostatic Field. When a condenser is being charged a current flows into it from the source of electromotive force. This involves the expenditure of a certain amount of energy, necessary to produce the required stresses and strains in the dielectric of the condenser. This energy is not converted into heat, and therefore lost, as in the case of metallic conduction: The energy is stored in the potential form in the dielectric, and can be returned to the circuit by reducing the voltage at the condenser terminals. With reference to the analogy shown in Fig. 12, the energy expended by the pump in straining the partition is stored in the partition by virtue of its elastic character, and exists coincident with the stresses and strains. It can be returned to the piston rod by allowing the piston to be moved by the elastic partition.

It is necessary in some cases to calculate the energy stored in an electrostatic field; or at least to represent the energy stored per cubic inch or centimeter of dielectric as a function of the stress F, strain D, and the permittivity κ, at the point under consideration.

Consider first the simplest case of a plate condenser (Fig. 10), and neglect the small amount of displacement occurring outside the space between the plates. Let the condenser be charged by gradually raising the voltage at its terminals from 0 to a final

value E; let e and i be the instantaneous values of the voltage and of the charging current at a moment t during the process of charging.* The total electrical energy delivered to the condenser in charging is

$$W = \int_0^T ei\,dt = \int_0^T e.dq \quad . \quad . \quad . \quad . \quad . \quad (78)$$

where T is the total time of charging, and $dq = idt$ is the infinitesimal charge or displacement added to the condenser during the interval of time dt. The quantities dq and e can be expressed through the instantaneous flux density D_t and the stress F_t from eqs. (59) and (60). Performing the substitution, and taking the constant quantities A and l outside the sign of integration, we get

$$W = Al \int_0^T F_t dD_t. \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

In order to integrate this expression D_t, must be expressed through F_t, or vice versa. The relation between the two is given by eq. (65). Eliminating D_t we obtain:

$$W = \kappa Al \int_0^T F_t dF_t = \tfrac{1}{2}\kappa v F^2 \quad . \quad . \quad . \quad . \quad . \quad (80)$$

where $v = Al$ is the volume of the dielectric, and F is the final value of the stress, at the time T. Hence, the energy stored per unit volume of dielectric is

$$W/v = \tfrac{1}{2}\kappa F^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

The ratio W/v is called the *density of energy*. Using the relation $D = \kappa F$, eq. (81) can also be written in the following two forms:

$$W/v = \tfrac{1}{2} FD = \tfrac{1}{2} D^2/\kappa. \quad . \quad . \quad . \quad . \quad . \quad (82)$$

The analogy with formula (13) in Art. 4 is apparent at once.

The stored energy can also be expressed through the permittance or capacity of the dielectric. We have from eq. (68) $dq = C.de$; substituting into (78) and integrating we get

$$W = \tfrac{1}{2} CE^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

*The voltage aud the charging current rise gradually, even though the key K is closed suddenly. This is on account of an ever-present inductance in the leads which inductance acts as a kind of electromagnetic inertia.

Since the final charge, or total displacement, $Q = CE$, the energy can be also represented in the following two forms:

$$W = \tfrac{1}{2} QE = \tfrac{1}{2} Q^2/C. \quad . \quad . \quad . \quad . \quad . \quad (84)$$

These expressions are analogous to eq. (7a) in Art. 2.

Let now the condenser be of an irregular form, as shown in Fig. 13. The stress F and the displacement D are different at different points, so that it is necessary to consider infinitesimal layers of the dielectric between consecutive equipotential surfaces, and infinitesimal threads of displacement between the electrodes. Consider an infinitesimal volume $mnpq$ of the dielectric, comprised of a filament HH', between two equipotential surfaces MN and $M'N'$. The sides mp and nq can be provided with infinitely thin metal films, because these sides lie in the equipotential surfaces so that no current could flow along these metal coatings. Then the element of volume under consideration is converted into a small plate condenser; the flux density and the stress within this element can be considered uniform, so that formula (80) holds true, and we have

$$dW = \tfrac{1}{2} \kappa F^2 \cdot dv. \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

Differentials are used because both the volume and the energy stored are infinitesimal. The density of energy

$$dW/dv = \tfrac{1}{2} \kappa F^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

has the same expression as in the case of a plate condenser; but its numerical value is different from point to point, since F is variable. The two other expressions for the density of energy

$$dW/dv = \tfrac{1}{2} FD = \tfrac{1}{2} D^2/\kappa \quad . \quad . \quad . \quad . \quad (87)$$

also hold true for the points of a non-uniformly stressed dielectric, provided that proper values of D and F are used for each point.

The total energy stored in a non-uniform electrostatic field is equal to

$$W = \tfrac{1}{2} \int \kappa F^2 dv = \tfrac{1}{2} \int FD\, dv = \tfrac{1}{2} \int D^2 dv/\kappa. \quad . \quad . \quad (88)$$

Here F and D must be given as functions of co-ordinates, and the integration extended over the whole space occupied by the field. Eqs. (83) and (84) are true for condensers of any shape,

because in the deduction of these formulae no assumption was made as to the particular form of the dielectric, or of the electrodes.

The expressions for the electrostatic energy of the field, derived above, are identical with the expressions for the potential energy of stressed elastic bodies, and this is consistent with the assumed structure of the dielectric, shown in Fig. 11. Namely, consider the work necessary to strain the elastic fibres per one cubic centimeter of the material. During the process of shearing the stress varies from zero to its final value F. Let F_t be some intermediate value, and D_t the corresponding strain. While the strain increases from D_t to $(D_t + dD_t)$ the stress F_t may be considered constant; the infinitesimal work of shearing is therefore equal to $F_t dD_t$. The total work of shearing

$$W = \int_0^T F_t dD_t.$$

But, according to Hooke's law of elasticity, shearing strains are proportional to stresses, so that a relation exists between D and F, similar to eq. (65). We thus arrive again at the result that the work necessary to strain one cubic unit of an elastic material is equal to $\frac{1}{2}\kappa F^2$.

PROB. 136. Calculate the total stored energy, and the density of energy in the condenser given in problem 108. Ans. 20.52 milliwatt-seconds (millijoules); 19.53 microjoules per cubic centimeter.

PROB. 137. Assuming the relative permittivity of the insulation given in problem 110 to be 2.5; what is the density of energy at which the material is broken down? Ans. $\frac{1}{2}\kappa F^2 = .0725$ joules per cu. inch.

PROB. 138. Show that in a single-core cable (Fig. 14) the density of energy in the insulation varies inversely as the square of the distance from the center of the core.

PROB. 139. Show that in a spherical condenser the density of energy varies inversely as the fourth power of the distance from the center.

PROB. 140. Check in problem 111 that the energy supplied to the cable from the source is equal to that stored in the dielectric. The specific inductive capacity of the insulation is 2.73 (see

answer to problem 122). Solution: Max. density of energy, $(dW/dv)_{max} = \frac{1}{2} \times (.467 \times 10^{-6})^2/(.2244 \times 10^{-12} \times 2.73) = 17.8 \times 10^{-4}$ joules per cu. inch. The density of energy at a distance x from the center is $17.8 \times 10^{-4} (a_1/x)^2$, since the energy density varies inversely as the square of the distance from the center. The energy in an infinitesimal concentric layer of thickness dx, and one inch long, is $dW = 17.8 \times 10^{-4} (a_1/x)^2 \cdot 2\pi x\, dx$. Integrating between the limits a_1 and a_2 gives the total energy per inch length of the cable, $W = 1.798 \times 10^{-4}$ joules. The energy per mile is 11.39 joules. On the other hand, according to eq. (84), the same energy is equal to $\frac{1}{2} \times .0019 \times 12000 = 11.4$ joules.

26. The Electrostatic Corona. When the voltage is raised sufficiently high at the terminals of an air-condenser, a pale violet light appears at the edges, at the sharp points, and in general at the protruding parts having a comparatively small radius of curvature. This silent discharge into air, due to an excessive electrostatic flux density, is called the *electrostatic corona*. In the regions, where the corona appears, the air is electrically "broken down" and ionized so that it becomes a conductor of electricity. When the voltage is raised still higher the so-called brush discharge takes place, until the whole thickness of the dielectric is broken down, and a *disruptive discharge*, or spark, jumps from one electrode to the other.

When the electrodes have projecting parts or sharp edges the corona is formed at a voltage far below that at which the disruptive discharge occurs; the operating voltage of such devices is generally limited to that at which the corona forms. No corona is usually permissible in regular operation, first, because it involves a considerable loss of power; second, because the discharge, if allowed to play on some other insulation, will soon char and destroy it. There are cases, however, in which some corona formation is harmless: when the air which is broken down becomes a part of the electrode, smooths down the shape of the protruding parts, increases their area, and thus reduces the dangerous flux density, making it more uniform.

The formation of corona must be kept in mind in the design of high-tension insulation, and in high-potential tests. Shapes and combinations of parts should be avoided which lead to high

or non-uniform dielectric flux densities. Fig. 13 shows the reason why the dielectric flux density or the potential gradient is higher near protruding parts. The equipotential surfaces, for obvious geometrical reasons, lie closer to each other near such parts, because at a reasonable distance from the electrodes the shape of the equipotential surfaces is not affected by small irregularities in the shape of the metallic parts.

Corona is formed, in other words air is broken down, at a dielectric flux density D_{max} of about .00238 microcoulombs per square centimeter, or .0153 mc. per sq. inch (A. C., effective value, sine-wave voltage). This flux density corresponds to a potential gradient or stress $F_{max} = D_{max}/\kappa = 26.8$ effective kilovolts per centimeter (68.1 kv. per inch). In practice, it is advisable to keep somewhat below these figures.

PROB. 141. A conductor 1 inch in diameter is surrounded by a concentric metal cylinder, 6 inch, inside diameter. What alternating voltage can be allowed between the cylinder and the conductor at a factor of safety of 1.5 against the formation of the corona? Solution: The permissible flux density at the surface of the conductor is $.0153/1.5 = .01023$ mc. per sq. inch. Hence, the charge per inch of axial length must not exceed $\pi \times 1 \times .01023 = .0322$ mc. According to formula (68), the voltage is equal to the charge divided by the permittance of the dielectric. The permittance is calculated in this case according to eq. (73) and is equal to $.7865 \times 10^{-6}$ m.f. per inch length. Consequently, the permissible voltage

$$E = \left[.0322/(.7865 \times 10^{-6}) \right] 10^{-3} = 40.8 \text{ kv.}$$

PROB. 142. Show that in the preceding problem the maximum permissible voltage is proportional to the expression $a_1 \log (a_2/a_1)$, see Fig. 14. Plot a curve giving values of maximum permissible voltage, against diameters of the inner conductor as abscissae. The diameter of the outside cylinder is assumed constant and equal to 6 inches. Ans. The curve reaches a maximum = 50.3 kv. when the diameter of the conductor is 2.21 inch.

PROB. 143. The charging current per mile of wire in a transmission line is .1 amp. Given that the conductors consist of No. .0000 wire B. & S. and that the frequency is 25 cycles per second, determine the factor of safety of the line against the corona.

Solution: The line is charged during one quarter of a cycle, or during 1/100 of a second. The average current is $(.1 \times \sqrt{2}) \times (2/\pi) = .09$ amp. Maximum charge is $.09 \times (1/100) \times 10^6 = 900$ mc. per mile. The area of the wire per mile is 91600 sq. inch., so that the flux density is $900/91600 = .00983$ mc. per sq. inch. The factor of safety is $.0153/.00983 = 1.56$.

27. Dielectric Hysteresis and Conduction. When an alternating voltage is applied at the terminals of a condenser, the dielectric is subjected to periodic stresses and strains. If the material were perfectly elastic, no energy would be lost during one complete cycle, because the energy stored during the periods of increase in voltage would be given up to the circuit when the voltage decreased. In reality, there are reasons to believe that the electric elasticity of solid and liquid dielectrics is not perfect. Referring to Fig. 11, dielectrics behave as if there was some kind of friction between the filaments, so that the applied voltage has to overcome this friction, in addition to the elastic forces. The work done against friction is converted into heat, and is lost, as far as the circuit is concerned. The phenomenon is similar to the familiar magnetic hysteresis, and is therefore called *dielectric hysteresis.* The energy lost per cycle is proportional to the square of the applied voltage, because both the displacement and the stress are proportional to the voltage.

Under normal conditions, that is to say when stresses are well below the ultimate stress, the loss of power in dielectric hysteresis is exceedingly small. Some investigators even doubt its existence at all. There is often an appreciable loss of power in commercial condensers, but this loss can be mostly attributed to the fact that dielectrics are not perfect insulators. While their ohmic resistance is exceedingly high, as compared to that of metals, they nevertheless conduct some current, especially at high voltages. Thus, the observed loss of power, and the heating of condensers, can be simply ascribed to the I^2R loss in the insulation. Moreover, small amounts of corona can form at the edges and projecting parts, even at the operating voltage, and thus be an additional source of loss. Some small loss is also due to the ohmic resistance of the metallic sheets which compose the condenser.

An imperfect condenser, that is to say a condenser which shows a loss of power, from one cause or another, can be replaced, for purposes of calculation, by a perfect condenser with an ohmic resistance shunted around it. The resistance is selected of such a value that the I^2R loss in it is equal to the loss of power from all the causes in the given imperfect condenser. The actual current through the imperfect condenser is considered then as consisting of two components: The wattless component, flowing through the ideal condenser, and the loss component, in phase with the voltage, flowing through the shunted resistance. In this way, imperfect condensers can be treated graphically or analytically, according to the ordinary laws of the electric circuit.

PROB. 144. A certain kind of condensers shows a loss of power of about 17.9 watts per microfarad, at about 2200 v., 25 cycles. By what fictitious resistance should an ideal condenser be shunted in order to replace a condenser of this kind, having a capacity of 1.5 mf.? Ans. About .18 megohm.

LITERATURE.

Alternating Current Phenomena, by Chas. P. Steinmetz.

Theoretical Elements of Electrical Engineering, by Chas. P. Steinmetz.

Alternating Currents, by Bedell and Crehore.

Vectors and Vector Diagrams, by Cramp and Smith.

Electrical Engineering, by Thomälen.

Die Wissenschaftlichen Grundlagen der Elektrotechnik, by Benischke.

Die Wechselstromtechnik, Vol. I, by Arnold and la Cour.

Problems in Electrical Engineering, by Waldo V. Lyon.

Electrical Problems, by Hooper and Wells.

The Elements of Electrical Engineering, Vol. II, by Franklin and Esty.

Modern Views of Electricity, by Oliver Lodge.

Electric Waves, by W. S. Franklin.

Elements of Electromagnetic Theory, by S. J. Barnett.

Kapazität und Induktivität, by Ernst Orlich.

Text-book of Electrical Machinery, by Ryan, Norris, and Hoxie.

·4 [illegible]
150 net ×